Landau's Theory

Universality $\Rightarrow$ There may be a simple theory valid near a critical point that can QUANTITATIVELY explain the vanishing of the order parameter and the divergences of the heat capacity, susceptibility and the correlation length.

Large correlation length $\Rightarrow$ Microscopic details may not be important and a COARSE GRAINED description may give us the correct answers.

Approach – Write down the simplest theory that is consistent with the symmetries of the problem being studied.
– Evaluate macroscopic quantities of interest near the critical point.
– Modify the theory if necessary.

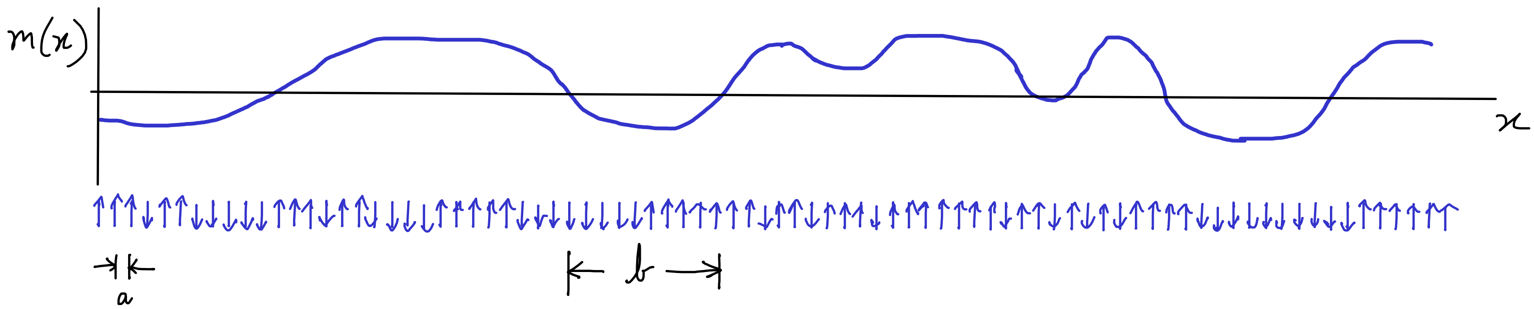
To get the properties of a system near a critical point, we need to calculate the partition function near the critical point.

$Z = \mathrm{Tr}\, e^{-\beta H}$, where H is the microscopic Hamiltonian and depends on the microscopic degrees of freedom of the system.
This calculation is extremely difficult and cannot be carried out exactly.

Coarse grained/mesoscopic description

Define a local order parameter $m(\vec{x})$.
This is obtained by suitably averaging over large numbers of $\underbrace{\text{microscopic variables}}_{\text{positions / momenta of particles, spins, etc}}$ around each location $\vec{x}$.

Thus, the fluctuations of $m(\vec{x})$ are much smaller than the fluctuations of the microscopic variables.
Also, the lengthscale over which $m(\vec{x})$ varies is much larger than the microscopic length (e.g. lattice constant, interatomic spacing, etc).



a = microscopic length

b = mesoscopic length $\qquad a \ll b \ll L$

L = system size

At the critical point, $m(\vec{x}) = 0$. Near the critical point $m(\vec{x})$ is small.

$m(\vec{x})$ can be a vector with n- components (depending on what problem we would like to describe).

$n = 1$ (scalar) $\rightarrow$ Liquid-gas transition, uniaxial magnets.

$n = 2$ $\rightarrow$ Planar magnets (XY-model), superfluidity, superconductivity.

$n = 3$ $\rightarrow$ Conventional magnets

Here, for simplicity, we will focus on problems that can be described by one component local order parameters.

Let us assume that

$$Z = \mathrm{Tr}\, e^{-\beta H} = \mathop{\mathrm{Tr}}_{\{m(\vec{x})\}} e^{-\beta H_{eff}[m(\vec{x})]}$$

$H_{eff}[m(\vec{x})]$ is a functional of $m(\vec{x})$.

$$\mathop{\mathrm{Tr}}_{\{m(\vec{x})\}} e^{-\beta H_{eff}[m(\vec{x})]} = \int \mathcal{D}m(\vec{x})\, e^{-\beta H_{eff}[m(\vec{x})]},$$

where $\int \mathcal{D}m(\vec{x}) = \prod_{\vec{x}} \int_{-\infty}^{\infty} dm(\vec{x})$

Let $\beta H_{eff}[m(\vec{x})] = \int d^d x\, \Psi$,

where Ψ is known as the Landau free energy.

What does Ψ depend on?

- Ψ must depend on $m(\vec{x})$.
- It can depend on an external field which could be position dependent — $h(\vec{x})$.
- An inhomogeneous phase ($m(\vec{x})$ varying in space) must have a lower energy cost than a homogeneous phase. This is captured by a term such as $\frac{K}{2}|\nabla m|^2$, where $K>0$.
- When $h(\vec{x})=0$, the sign of $m(\vec{x})$ must not be important. This suggests terms such as $\frac{1}{2}r_0 m^2(\vec{x}) + u_0 m^4(\vec{x})$.
- When $h(\vec{x}) \neq 0$, it should align the local order parameter in its direction. This is achieved by terms such as $-\beta\, m(\vec{x})\, h(\vec{x})$.

Keeping all these things in mind, the simplest Landau free energy can be written as,

$$\Psi\left(m(\vec{x}), \nabla m(\vec{x}), h(\vec{x})\right) = \frac{K}{2}|\nabla m(\vec{x})|^2 - \beta\, m(\vec{x})\, h(\vec{x}) + \frac{1}{2} r_0 m^2(\vec{x}) + u_0 m^4(\vec{x})$$

[If required higher powers of $m(\vec{x})$ and $\nabla m(\vec{x})$ can be inserted.]

$\{K, r_0, u_0\}$ are functions of the reduced temperature t.

To be able to explain critical phenomena, we require r_0 to vanish at the critical point.

Thus, keeping lowest order terms in the Taylor series expansion of $\{K, r_0, u_0\}$ as functions of t, we have,

$r_0 = a_0 t \qquad (a_0 = \text{constant})$

$u_0 = \text{constant}$

$K = \text{constant}$ (chosen to be 1. This sets the scale of $m(\vec{x})$)

$$\therefore \Psi\left(m(\vec{x}), \nabla m(\vec{x}), h(\vec{x})\right) = \frac{1}{2}|\nabla m(\vec{x})|^2 - \beta\, m(\vec{x})\, h(\vec{x}) + \frac{a_0 t}{2} m^2(\vec{x}) + u_0 m^4(\vec{x})$$

Saddle point approximation / Mean Field Theory

When we calculate the partition function,

$$Z = \int \mathcal{D}m(\vec{x})\, e^{-\beta H_{eff}[m(\vec{x})]},$$

the dominant contribution comes from the function, $m(\vec{x})$, for which H_{eff} is the smallest.

To satisfy the first term in Ψ, the optimal $m(\vec{x})$ must not have spatial variations, i.e. $m(\vec{x}) = m =$ constant.

Also, $\dfrac{\partial \Psi}{\partial m} = 0$ and $\dfrac{\partial^2 \Psi}{\partial m^2} > 0$

Let's focus on the situation with a uniform field, $h(\vec{x}) = h$.

$$\Rightarrow \Psi(m, h) = -\beta m h + \frac{a_0 t}{2} m^2 + u_0 m^4$$

$$\Rightarrow \frac{\partial \Psi}{\partial m} = -\beta h + a_0 t m + 4u_0 m^3 = 0 \quad \text{--- (1)}$$

If $h = 0$, $m = 0$ for $t > 0$

$$m = \pm \sqrt{\frac{a_0}{4u_0}}\, |t|^{1/2} \quad \text{for } t < 0$$

$\Rightarrow \therefore m \sim |t|^{\beta}$, $\boxed{\beta = 1/2}$. The β before this line represents $1/k_B T$. The distinction between the two betas should be clear from context.

Differentiating Eq. (1) w.r.t. h, we get,

$$-\beta + a_0 t \chi + 12 u_0 m^2 \chi = 0 \qquad \chi = \left.\frac{\partial m}{\partial h}\right|_{h=0}$$

$$\Rightarrow \chi = \frac{1}{k_B T (a_0 t + 12 u_0 m^2)}$$

$$= \begin{cases} \dfrac{1}{k_B T a_0 t}, & \text{for } t > 0 \\[2ex] \dfrac{1}{k_B T (a_0 t + 12 u_0 m^2)} = \dfrac{1}{k_B T \left(-a_0 |t| + 12 u_0 \frac{a_0}{4u_0} |t|\right)} = \dfrac{1}{2 k_B T a_0 t}, & \text{for } t < 0. \end{cases}$$

$$\Rightarrow \therefore \chi \sim |t|^{-\gamma}, \quad \boxed{\gamma = 1}$$

For $t=0$, from Eq. (1),

$$-\beta h + 4u_0 m^3 = 0 \Rightarrow m = \left(\frac{1}{4u_0 k_B T}\right)^{1/3} h^{1/3}$$

$$\Rightarrow \therefore m \sim h^{1/\delta} \quad \boxed{\delta = 3}$$

To get the internal energy, we go back to the partition function.

$$Z = \int \mathcal{D}m(\vec{x})\, e^{-\int d^d x\, \Psi}$$

$$\propto e^{-V\left(-\beta m h + \frac{a_0 t}{2} m^2 + u_0 m^4\right)}$$

If $h=0$, then, $Z = e^{-V\left(\frac{a_0 t}{2} m^2 + u_0 m^4\right)}$.

$$U = -\frac{\partial}{\partial \beta} \ln Z = \frac{\partial t}{\partial \beta} \frac{\partial}{\partial t}\left[V\left(\frac{a_0 t}{2} m^2 + u_0 m^4\right)\right]$$

$$= \frac{1}{T_c}\left(-\frac{1}{k_B \beta^2}\right) \frac{\partial}{\partial t}\left[V\left(\frac{a_0 t}{2} m^2 + u_0 m^4\right)\right]$$

$$= -\frac{k_B T^2}{T_c} \frac{\partial}{\partial t}\left[V\left(\frac{a_0 t}{2} m^2 + u_0 m^4\right)\right]$$

$t = \frac{T - T_c}{T_c}$

$T = \frac{1}{k_B \beta}$

For $t > 0$, $m = 0 \Rightarrow U = 0$

For $t < 0$, $U = -k_B T_c \frac{\partial}{\partial t}\left[V\left(\frac{a_0 t}{2} \frac{a_0}{4u_0}(-t) + u_0 \frac{a_0^2}{16 u_0^2} t^2\right)\right]$

$$= k_B T_c \frac{\partial}{\partial t}\left[\frac{V a_0^2 t^2}{16 u_0}\right]$$

$$= \frac{V a_0^2 k_B T_c}{8 u_0} t$$

In addition to the expression obtained above, U could have a regular part that is proportional to TV and is well-behaved at the critical point.

Taking this into account,

$$U = \begin{cases} C_0 V T + \dfrac{V a_0^2 k_B T_c}{8 u_0} t \;, & \text{for } t<0 \\ C_0 V T \;, & \text{for } t>0 \end{cases}$$

$\therefore$ The heat capacity per unit volume,

$$c = \frac{C}{V} = \frac{1}{V}\left(\frac{\partial U}{\partial T}\right)_V = \begin{cases} C_0 + a_0^2 k_B / 8 u_0 \;, & \text{for } t<0 \\ C_0 \;, & \text{for } t>0 \end{cases}$$

Thus, in Landau's theory, we get a discontinuity in the heat capacity but no divergence.

$$\Rightarrow \boxed{\alpha = 0}$$

To understand the behavior of the correlation, we must go beyond the uniform phase and include fluctuations.