Sterling's approximation:

$$x! \sim \sqrt{2\pi x}\, x^{x} e^{-x}, \quad \text{as} \quad x \to \infty$$

x	$x!$	Sterling	Error %
0	1	0.00	100
1	1	0.92	7.79
2	2	1.92	4.05
3	6	5.84	2.73
4	24	23.51	2.06
5	120	118.02	1.65

Note that this is the first term of the Sterling series, which gives an even better approximation to $x!$ when more terms are used.

$$N_+ + N_- = N$$

$$\binom{N}{N_+} \sim \frac{\sqrt{2\pi N}\; N^{N} e^{-N}}{\sqrt{2\pi N_+}\; N_+^{N_+} e^{-N_+} \sqrt{2\pi N_-}\; N_-^{N_-} e^{-N_-}}$$

$$= \frac{\sqrt{N}\quad N^{N}}{\sqrt{2\pi}\;\frac{N}{2}\sqrt{1-L^2}\left(\frac{N}{2}(1+L)\right)^{N_+}\left(\frac{N}{2}(1-L)\right)^{N_-}}$$

$$= \frac{2^{N+1/2}}{\sqrt{\pi N}\,\sqrt{1-L^2}}\;\frac{1}{(1-L^2)^{N/2}}\left(\frac{1-L}{1+L}\right)^{NL/2}$$

$$= \frac{2^{N+1/2}}{\sqrt{\pi N}\,\sqrt{1-L^2}}\exp\left[-\frac{N}{2}\ln(1-L^2) + \frac{NL}{2}\ln(1-L) - \frac{NL}{2}\ln(1+L)\right]$$

$$\Rightarrow Z = \frac{2^{N+1/2}}{\sqrt{\pi N}}\sum_{L}\frac{1}{\sqrt{1-L^2}}\exp\left[\beta N\left(\frac{J\gamma L^2}{2} + hL\right) - \frac{N}{2}\ln(1-L^2) + \frac{NL}{2}\ln(1-L) - \frac{NL}{2}\ln(1+L)\right]$$

$$= \frac{2^{N+1/2}}{\sqrt{\pi N}}\sum_{L} g(L)\, e^{-Nf(L)}$$

$$\sim \frac{2^{N+1/2}}{\sqrt{\pi N}}\, g(m)\, e^{-Nf(m)}, \quad \text{as} \quad N \to \infty$$

where m is the value of L that minimizes $f(L)$.

Here, $g(L) = \dfrac{1}{\sqrt{1-L^2}}$, and,

$$f(L) = -\frac{\beta J\gamma L^2}{2} - \beta hL + \frac{1}{2}\ln(1-L^2) - \frac{L}{2}\ln(1-L) + \frac{L}{2}\ln(1+L)$$

$$f'(m) = 0$$

$$\Rightarrow -\beta J\gamma m - \beta h - \frac{m}{1-m^2} - \frac{1}{2}\ln(1-m) + \frac{m/2}{1-m} + \frac{1}{2}\ln(1+m) + \frac{m/2}{1+m} = 0$$

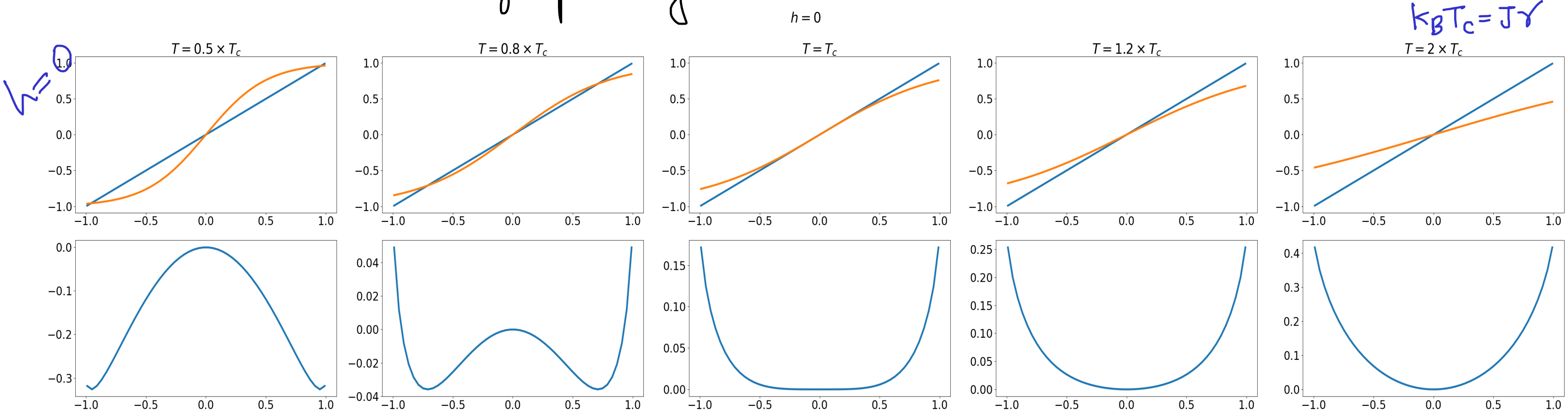
$$\Rightarrow \quad \beta(J\gamma m + h) = \frac{1}{2}\ln\left(\frac{1+m}{1-m}\right)$$

$$\Rightarrow \quad 1+m = (1-m)\, e^{2\beta(J\gamma m + h)}$$

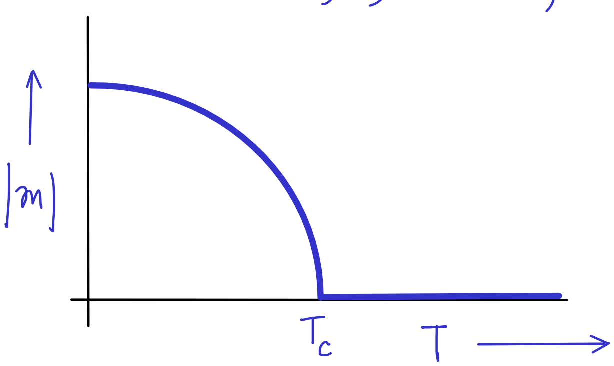
$$\Rightarrow \quad m\left[1 + e^{2\beta(J\gamma m+h)}\right] = 1 - e^{2\beta(J\gamma m+h)}$$

$$\Rightarrow \quad \boxed{m = \tanh\left[\beta(J\gamma m + h)\right]}$$

This can be solved graphically.



Top line: $y = x$ and $y = \tanh(\beta J\gamma x)$. Intersections give the values of m.
Bottom line: $f(L)$ vs L, showing that $L = m$ minimizes $f(L)$.



$T < T_c$: $|m| > 0 \Rightarrow$ Ferromagnet

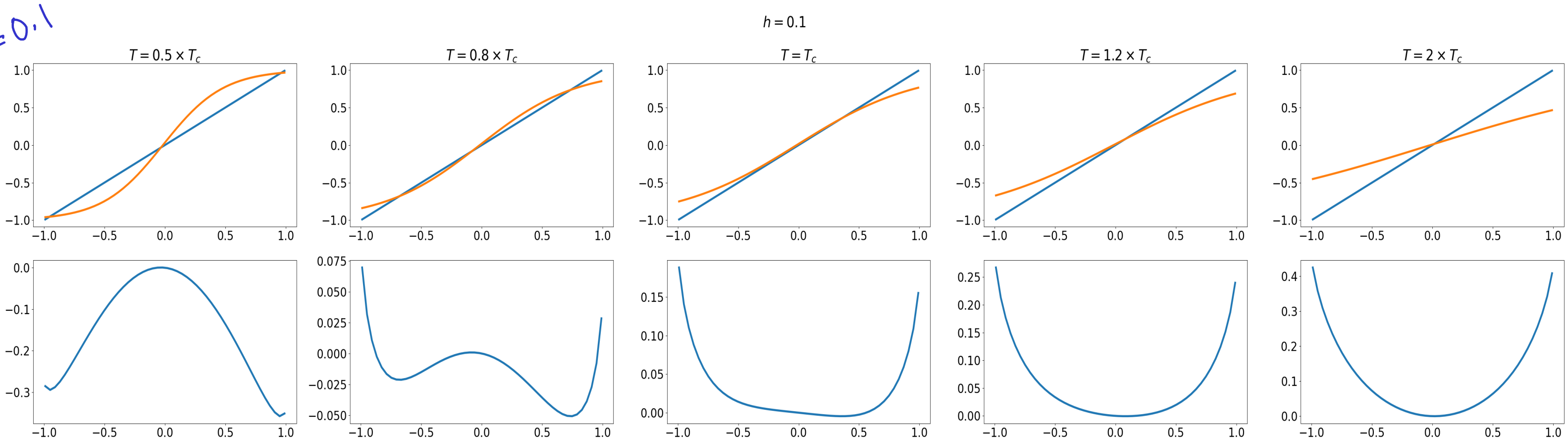
$T > T_c$: $|m| = 0 \Rightarrow$ Paramagnet

For small x, $\tanh(\beta J\gamma x) \sim \beta J\gamma x$

$\Rightarrow$ At T_c, the slope near zero must be 1

$\Rightarrow$ $T_c = J\gamma / k_B$

$h=0.1$



Let's go back to the partition function.

$$Z \simeq \frac{2^{N+1/2}}{\sqrt{\pi N}}\, g(m)\, e^{-Nf(m)}$$

$$f(m) = -\frac{\beta J\gamma m^2}{2} - \beta h m + \frac{1}{2}\ln(1-m^2) - \frac{m}{2}\ln(1-m) + \frac{m}{2}\ln(1+m)$$

Since $\beta(J\gamma m + h) = \frac{1}{2}\ln\left(\frac{1+m}{1-m}\right)$,

$$f(m) = -\frac{\beta J\gamma m^2}{2} - \beta h m + \frac{1}{2}\ln(1-m^2) + m\beta(J\gamma m + h)$$

$$= \frac{\beta J\gamma m^2}{2} + \frac{1}{2}\ln(1-m^2)$$

The free energy per site is,

$$\frac{F}{N} = -\frac{k_B T}{N}\ln Z \simeq k_B T\left(f(m) - \ln 2\right) \quad \text{as } N\to\infty$$

$$\Rightarrow \frac{F}{N} = \frac{J\gamma m^2}{2} + \frac{k_B T}{2}\ln\left(\frac{1-m^2}{4}\right)$$

The magnetization per site is,

$$\frac{M}{N} = -\frac{\partial(F/N)}{\partial h} = -\left(J\gamma m - \frac{k_B T\, m}{1-m^2}\right)\frac{\partial m}{\partial h} \quad ---(1)$$

Using $m = \tanh\left(\beta(J\gamma m + h)\right)$, we get,

$$\frac{\partial m}{\partial h} = \underbrace{\operatorname{sech}^2\left(\beta(J\gamma m + h)\right)}_{1-\tanh^2\left(\beta(J\gamma m+h)\right) = 1-m^2}\left(\beta J\gamma\frac{\partial m}{\partial h} + \beta\right)$$

$$\Rightarrow \quad \frac{\partial m}{\partial h} = \frac{(1-m^2)\beta}{1-(1-m^2)\beta J\gamma} = \frac{1}{\frac{k_BT}{1-m^2} - J\gamma}$$

$\Rightarrow$ Using Eq. (1), $\boxed{\frac{M}{N} = m}$

When $h=0$, approaching T_c from lower temperatures, m vanishes at $T_c \Rightarrow m$ is small near T_c, i.e. we focus on the conditions, $h=0$, $k_B(T_c-T) \ll J \Rightarrow m \ll 1$ but $m \neq 0$.

$$\Rightarrow \frac{F}{N} \simeq \frac{J\gamma m^2}{2} - \frac{k_BT}{2}m^2 - k_BT\ln 2 = k_B(T_c-T)\frac{m^2}{2} - k_BT\ln 2$$

$$\Rightarrow \frac{U}{N} = -\frac{\partial}{\partial\beta}\left(\frac{1}{N}\ln Z\right) = \frac{\partial}{\partial\beta}\left(\beta F/_N\right)$$

$$= \frac{\partial}{\partial\beta}\left[\left(\beta k_BT_c - 1\right)\frac{m^2}{2} - \ln 2\right]$$

$$= k_BT_c\frac{m^2}{2} + \left(\beta k_BT_c - 1\right) m \frac{\partial m}{\partial\beta}$$

$$m = \tanh(\beta J\gamma m)$$

$$\Rightarrow \frac{\partial m}{\partial\beta} = \underbrace{\operatorname{sech}^2(\beta J\gamma m)}_{1-m^2}\left(J\gamma m + \beta J\gamma\frac{\partial m}{\partial h}\right)$$

$$\Rightarrow \frac{\partial m}{\partial\beta} = \frac{(1-m^2)J\gamma m}{1-(1-m^2)\beta J\gamma} \simeq \frac{J\gamma m}{1-\beta J\gamma} = \frac{k_BT_c}{1-\beta k_BT_c}m$$

$$\therefore \frac{U}{N} = k_BT_c\frac{m^2}{2} - m^2k_BT_c = -k_BT_c\frac{m^2}{2}$$

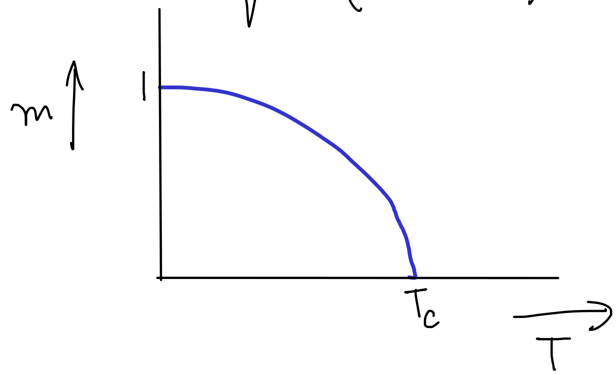
$$m = \tanh(\beta J\gamma m) \simeq \beta J\gamma m - \frac{1}{3}(\beta J\gamma m)^3$$

$$\Rightarrow m^2 \simeq \frac{3(\beta J\gamma - 1)}{(\beta J\gamma)^3} \qquad \because m \neq 0$$

$$\Rightarrow m \simeq \frac{3}{(\beta J\gamma)^2}\left(1 - \frac{T}{T_c}\right) \qquad \because J\gamma = k_BT_c$$

As $T \to T_c$, $\quad \beta J \gamma \to 1$

$$\Rightarrow \quad m \simeq \sqrt{3\left(1 - T/T_c\right)} \qquad \text{for } T \to T_c^-$$



$$\Rightarrow \quad \frac{U}{N} \simeq -\frac{k_B T_c}{2} m^2 \simeq \frac{3}{2} k_B \left(T - T_c\right)$$

$\Rightarrow$ Heat capacity, $\quad C = \frac{3}{2} N k_B \qquad$ for $T \to T_c^-$

When $h = 0$, $\quad m = 0$ for $T > T_c$.

$$\Rightarrow \quad \frac{M}{N} = m = 0$$

and $\quad f(m) = 0, \quad g(m) = 1$

$$\Rightarrow \quad Z = \frac{2^{N+1/2}}{\sqrt{\pi N}}$$

$$\Rightarrow \quad \frac{F}{N} = -\frac{k_B T}{N} \ln Z \simeq -k_B T \ln 2$$

$$\Rightarrow \quad \frac{U}{N} = \frac{\partial\left(\beta F/N\right)}{\partial \beta} = 0 \qquad \Rightarrow \quad \frac{C}{N} = 0$$

<u>Magnetic Susceptibility</u> $\qquad \chi = \lim_{h \to 0} \frac{\partial M}{\partial h} = N \lim_{h \to 0} \frac{\partial m}{\partial h}$

$$\Rightarrow \quad \chi = \frac{\beta N \left(1 - m^2\right)}{1 - \beta J \gamma \left(1 - m^2\right)} = \frac{N\left(1 - m^2\right)}{k_B\left[T - T_c\left(1 - m^2\right)\right]}$$

For $T > T_c$, $m \to 0$ as $h \to 0$ $\quad \Rightarrow \quad \chi = \frac{\beta N}{1 - \beta J \gamma} = \frac{N}{k_B\left(T - T_c\right)}$

For $T \to T_c^-$, $\quad m^2 \simeq 3\left(1 - T/T_c\right)$

$$\Rightarrow \chi \simeq \frac{N(1-m^2)}{k_B\left[T - T_c\left(\frac{3T}{T_c} - 2\right)\right]} \simeq \frac{N}{2k_B(T_c - T)}$$