Two-spin correlation function

$$\langle s_i s_{i+r} \rangle = \frac{1}{Z} \sum_{s_1} \cdots \sum_{s_N} s_i s_{i+r}\, e^{\beta \sum_{i=1}^{N} \left(J s_i s_{i+1} + \frac{h}{2}(s_i + s_{i+1}) \right)}$$

$$= \frac{1}{Z} \sum_{s_1} \cdots \sum_{s_N} \langle s_1 | T | s_2 \rangle \cdots \langle s_{i-1} | T | s_i \rangle s_i \langle s_i | T | s_{i+1} \rangle \cdots$$

$$\langle s_{i+r-1} | T | s_{i+r} \rangle s_{i+r} \langle s_{i+r} | T | s_{i+r+1} \rangle \cdots \langle s_N | T | s_1 \rangle$$

$$= \frac{1}{Z} \mathrm{Tr}\left(T^{i-1} \sigma_z T^{r} \sigma_z T^{N-i-r+1} \right)$$

$$= \frac{1}{Z} \mathrm{Tr}\left(S S^{-1} T^{i-1} S S^{-1} \sigma_z S S^{-1} T^{r} S S^{-1} \sigma_z S S^{-1} T^{N-i-r+1} S S^{-1} \right)$$

$$= \frac{1}{Z} \mathrm{Tr}\left(S \Lambda^{i-1} S^{-1} \sigma_z S \Lambda^{r} S^{-1} \sigma_z S \Lambda^{N-i-r+1} S^{-1} \right)$$

$$= \frac{1}{Z} \mathrm{Tr}\left(\Lambda^{r} S^{-1} \sigma_z S \Lambda^{N-i-r+1} S^{-1} S \Lambda^{i-1} S^{-1} \sigma_z S \right)$$ using cyclic property

$$= \frac{1}{Z} \mathrm{Tr}\left(\Lambda^{r} S^{-1} \sigma_z S \Lambda^{N-r} S^{-1} \sigma_z S \right)$$

$$= \frac{1}{Z} \mathrm{Tr}\left(\begin{pmatrix} \lambda_+^r & 0 \\ 0 & \lambda_-^r \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_+^{N-r} & 0 \\ 0 & \lambda_-^{N-r} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

$$= \frac{1}{Z} \mathrm{Tr}\left(\begin{pmatrix} 0 & \lambda_+^r \\ \lambda_-^r & 0 \end{pmatrix} \begin{pmatrix} 0 & \lambda_+^{N-r} \\ \lambda_-^{N-r} & 0 \end{pmatrix} \right)$$

$$= \frac{1}{Z} \mathrm{Tr}\begin{pmatrix} \lambda_+^r \lambda_-^{N-r} & 0 \\ 0 & \lambda_+^{N-r} \lambda_-^r \end{pmatrix}$$

$$= \frac{\lambda_+^r \lambda_-^{N-r} + \lambda_+^{N-r} \lambda_-^r}{\lambda_+^N + \lambda_-^N}$$

$$\langle s_i s_{i+r} \rangle = \frac{\left(\frac{\lambda_-}{\lambda_+}\right)^r + \left(\frac{\lambda_-}{\lambda_+}\right)^{N-r}}{1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N} \sim \left(\frac{\lambda_-}{\lambda_+}\right)^r \quad \text{as} \quad N \to \infty$$

$$\Rightarrow \langle s_i s_{i+r} \rangle \sim \exp\left(-r/\xi\right) \quad \text{as} \quad N \to \infty$$

where $\xi = \dfrac{1}{\ln(\lambda_+/\lambda_-)} = \dfrac{1}{\ln(\coth(\beta J))}$ $\begin{cases} \to 0 & \text{if } \beta J \to 0 \ (\text{High } T) \\ \to \infty & \text{if } \beta J \to \infty \ (\text{Low } T) \end{cases}$

Ising Model (arbitrary dimensions) in terms of long and short range order variables

$$L \equiv \frac{1}{N}\sum_i S_i = \frac{1}{N}\left(N_+ - N_-\right) \quad ---(1) \qquad \text{and} \qquad \sigma \equiv \frac{N_{++} - N_{bonds}/2}{N_{bonds}/2} \quad ---(2)$$

N_{++} = Number of bonds with up spins at both ends

$N_{bonds} = \gamma N/2$ = total number of bonds (γ = coordination number)

$$\gamma = \begin{cases} 2 & \text{for 1d chain} \\ 4 & \text{for 2d square lattice}, \ 6 \text{ for cubic lattice} \\ 2d & \text{for d-dimensional hypercubic lattice} \end{cases}$$

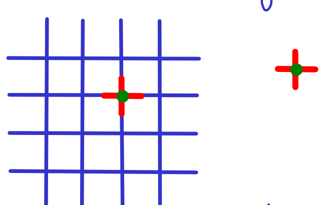
We know, $N_+ + N_- = N \quad ---(3)$

From (1) and (3), we have,

$$N_+ = \frac{N}{2}(1+L) \quad ---(4)$$

and $$N_- = \frac{N}{2}(1-L) \quad ---(5)$$

Let each bond be made of two half bonds that stick out of the sites. There are γ half-bonds coming out of each site



A site is coloured green and the four half bonds are coloured red.

The number of half bonds coming from all the up spin sites is

$$\gamma N_+ = 2N_{++} + N_{+-} \quad ---(6)$$

Similarly $$\gamma N_- = 2N_{--} + N_{+-} \quad ---(7)$$

From Eq. (2), $$N_{++} = \frac{\gamma N}{4}(1+\sigma) \quad ---(8)$$

$\Rightarrow$ From Eqs. (6, 4 $$N_{+-} = \frac{\gamma N}{2}(1+L) - \frac{\gamma N}{2}(1+\sigma)$$

$$\Rightarrow \quad N_{+-} = \frac{\gamma N}{2}(L-\sigma) \quad ---(9)$$

Since $N_{++} + N_{--} + N_{+-} = N_{bonds} = \gamma N/2$,

$$N_{--} = \frac{\gamma N}{2} - \frac{\gamma N}{4}(1+\sigma) - \frac{\gamma N}{2}(L-\sigma)$$

$$\Rightarrow \quad N_{--} = \frac{\gamma N}{4}(1+\sigma-2L)$$

$$\therefore \quad H = -J\sum_{\langle i,j\rangle} s_i s_j - h\sum_i s_i$$

$$= -J\left(N_{++} + N_{--} - N_{+-}\right) - h\left(N_+ - N_-\right)$$

$$= -JN_{bonds} + 2JN_{+-} - hLN$$

$$= -J\gamma N/2 + J\gamma N(L-\sigma) - hLN$$

$$\boxed{H = -\frac{J\gamma N}{2}\left(2\sigma - 2L + 1\right) - hLN}$$

$\therefore$ H depends only on L and σ.

Bragg-Williams approximation (Mean Field Theory)

Assumption : No additional short range order than what can be guessed from the long range order.

$$\begin{pmatrix}\text{Probability of finding} \\ \text{a } {+}{+} \text{ bond}\end{pmatrix} = \begin{pmatrix}\text{Square of the probability of} \\ \text{finding a } + \text{ bond}\end{pmatrix}$$

$$\Rightarrow \quad \frac{N_{++}}{N_{bonds}} = \left(\frac{N_+}{N}\right)^2$$

$$\Rightarrow \quad \frac{1+\sigma}{2} = \left(\frac{1+L}{2}\right)^2$$

$$\Rightarrow \quad L^2 + 2L + 1 = 2(1+\sigma)$$

$$\Rightarrow \quad L^2 = 2\sigma - 2L + 1$$

$$\therefore \quad \boxed{H_{MF} = -\frac{J\gamma N}{2}L^2 - hNL}$$

The total number of spin configurations that correspond to a given value of L $= \binom{N}{N_+} = \dfrac{N!}{N_+!\,N_-!}$, where $N_\pm = \frac{N}{2}(1 \pm L)$

$\therefore$ The partition function is,

$$Z = \mathrm{Tr}\, e^{-\beta H}$$

$$= \sum_L \binom{N}{N_+} \exp\left[\beta N\left(\frac{J\gamma L^2}{2} + hL\right)\right]$$