From last class,

$$\Rightarrow n\lambda^3 \approx \frac{4g}{3\sqrt{\pi}}\left[(\ln z)^{3/2} + \frac{\pi^2}{8}(\ln z)^{-1/2} + \cdots\right] \quad \text{---(1)}$$

$g = (2S+1)$ is the spin degeneracy
$g = 2$ for electrons

and
$$\frac{P\lambda^3}{k_B T} \approx \frac{8g}{15\sqrt{\pi}}\left[(\ln z)^{5/2} + \frac{5\pi^2}{8}(\ln z)^{1/2} + \cdots\right] \quad \text{---(2)}$$

$$\ln z \approx \frac{T_F}{T}$$

From Eq. (1),
$$n\lambda^3 \approx \frac{4g}{3\sqrt{\pi}}(\ln z)^{3/2}\left[1 + \frac{\pi^2}{8}(\ln z)^{-2} + \cdots\right]$$
$$\approx \frac{4g}{3\sqrt{\pi}}(\ln z)^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{T}{T_F}\right)^2 + \cdots\right]$$

$$\Rightarrow \ln z \approx \left(\frac{3n\lambda^3\sqrt{\pi}}{4g}\right)^{2/3}\left[1 + \frac{\pi^2}{8}\left(\frac{T}{T_F}\right)^2 + \cdots\right]^{-2/3}$$
$$\approx \left(\frac{T_F}{T}\right)\left[1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \cdots\right]$$

$$\Rightarrow \mu \approx \varepsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \cdots\right]$$

From Eq. (2),
$$\frac{P\lambda^3}{k_B T} \approx \frac{8g}{15\sqrt{\pi}}\left[(\ln z)^{5/2} + \frac{5\pi^2}{8}(\ln z)^{1/2} + \cdots\right]$$
$$\approx \frac{8g}{15\sqrt{\pi}}(\ln z)^{5/2}\left[1 + \frac{5\pi^2}{8}(\ln z)^{-2} + \cdots\right]$$
$$\approx \frac{8g}{15\sqrt{\pi}}\left(\frac{T_F}{T}\right)^{5/2}\left[1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \cdots\right]^{5/2}\left[1 + \frac{5\pi^2}{8}\left(\frac{T}{T_F}\right)^2 + \cdots\right]$$
$$\approx \frac{8g}{15\sqrt{\pi}}\left(\frac{T_F}{T}\right)^{5/2}\left[1 - \frac{5\pi^2}{24}\left(\frac{T}{T_F}\right)^2 + \cdots\right]\left[1 + \frac{5\pi^2}{8}\left(\frac{T}{T_F}\right)^2 + \cdots\right]$$

$$\Rightarrow P \approx \frac{8g}{15\sqrt{\pi}}\left(\frac{T_F}{T}\right)^{5/2}\frac{k_B T}{\lambda^3}\left[1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \cdots\right]$$
$$\approx P_F\left[1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \cdots\right]$$

where P_F is the Fermi pressure (the pressure of a Fermi gas at 0K).

$$P_F = \frac{8g}{15\sqrt{\pi}}\left(\frac{T_F}{T}\right)^{5/2} k_BT\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}$$

$$= \frac{g}{15\pi^2}\left(k_BT_F\right)^{5/2}\frac{(2m)^{3/2}}{\hbar^3}$$

$$\epsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3} \Rightarrow \epsilon_F^{3/2} = \frac{6\pi^2 n\hbar^3}{(2m)^{3/2}g}$$

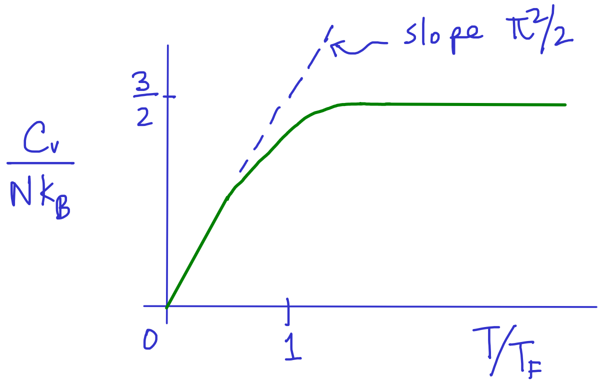
$$= \frac{\epsilon_F^{5/2}}{15\pi^2}\,\frac{6\pi^2 n}{\epsilon_F^{3/2}} \Rightarrow P_F = \frac{2}{5}n\epsilon_F \quad \text{(as seen before)}$$

$$\Rightarrow P \approx \frac{2}{5}n\epsilon_F\left[1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2+\dots\right]$$

$\therefore$ The internal energy is,

$$U = \frac{3}{2}PV = \frac{3}{5}N\epsilon_F\left[1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2+\dots\right]$$

$$\Rightarrow \text{Heat capacity, } C_V = \frac{\partial U}{\partial T} = \frac{3}{5}N\epsilon_F \times \frac{5\pi^2}{12}\times\frac{2T}{T_F} = \frac{\pi^2}{2}(Nk_B)\left(\frac{T}{T_F}\right)$$



Bose - Einstein condensation

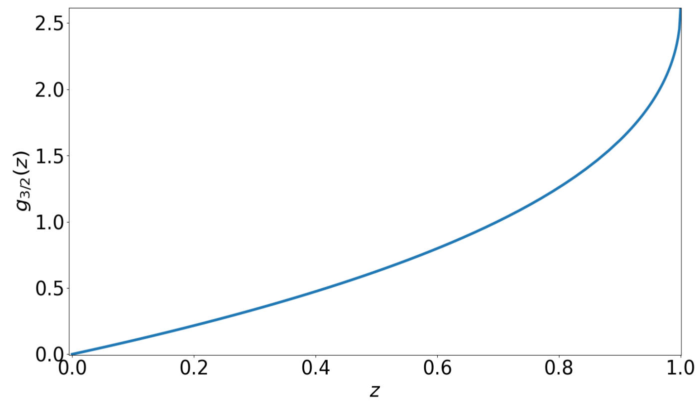
We know that $\bar{n}_{\vec{k}} = \dfrac{g}{e^{\beta(\epsilon_{\vec{k}}-\mu)}-1}$, $\quad g = \text{spin degeneracy} = 2S+1$

Since $\bar{n}_{\vec{k}}$ cannot be negative for any $\vec{k}$, $\mu \leq \min_{\vec{k}} \epsilon_{\vec{k}} = 0$ if $\epsilon_{\vec{k}} = \frac{\hbar^2k^2}{2m}$

$$\Rightarrow z = e^{\beta\mu} \leq 1$$

For a gas of bosons with conserved number of particles, $n\lambda^3 = g\, g_{3/2}(z)$.

where $g_{3/2}(z) = \dfrac{4}{\sqrt{\pi}}\displaystyle\int_0^\infty \frac{x^2\,dx}{\frac{1}{z}e^{x^2}-1} = \sum_{l=1}^\infty \frac{z^l}{l^{3/2}}$



Since $0 \leqslant z \leqslant 1$,

$z \to 0 \Rightarrow$ High T
$z = 1 \Rightarrow T \leqslant T_c$

$$g_{3/2}(z) \leqslant \sum_{l=1}^{k} \frac{1}{l^{3/2}} = \zeta(3/2) = 2.612$$

At high temperatures / low densities, for a given value of $n\lambda^3$, the value of z can be read off from the plot of $g_{3/2}(z)$.

E.g., if $n\lambda^3/g = 1$, $z \approx 0.7$.

As the temperature is lowered (or the density is increased), the value of the fugacity can be obtained from the relation $n\lambda^3 = g\, g_{3/2}(z)$, until, the LHS crosses the maximum value of the RHS.

When $n\lambda^3/g > \zeta(3/2)$, the value of z saturates to 1 and $\zeta(3/2)V/\lambda^3$ particles occupy states according to the Bose-Einstein distribution formula with $z=1$ ($\mu=0$). The remaining particles occupy the lowest available energy state ($\vec{k}=0$) and this state gets **macroscopically occupied**.

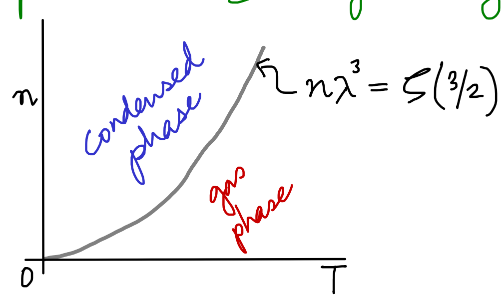
This phenomenon is known as the Bose-Einstein condensation.

The critical value of $n\lambda^3/g$ is $\zeta(3/2)$.

For a given density, the critical temperature T_c is given by,

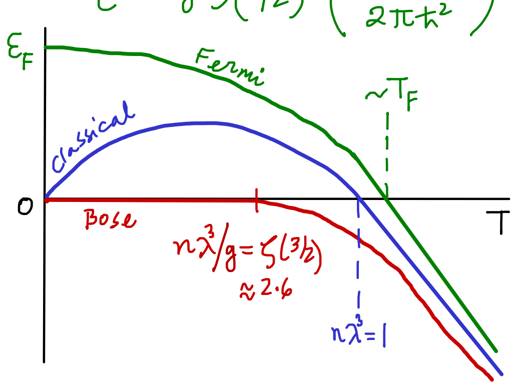
$$n\left(\frac{2\pi\hbar^2}{mk_BT_c}\right)^{3/2} = g\,\zeta(3/2)$$

$$\Rightarrow \quad T_c = \left[\left(\frac{1}{g\,\zeta(3/2)}\right)^{2/3} \frac{2\pi\hbar^2}{mk_B}\right] n^{2/3}$$



For a given temperature, the critical density n_c is given by

$$n_c = g\,\zeta(3/2)\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}$$



The condensate

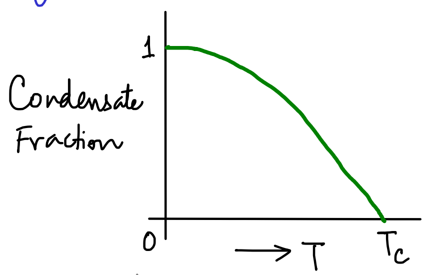
The number of particles that go into the condensate $(\vec{k}=0)$ state is

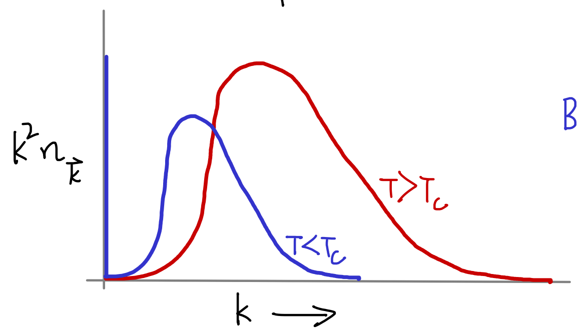
$$n_0 = \left[n - g\zeta(3/2)/\lambda^3\right]V$$

$\Rightarrow$ Fraction of particles in the condensate $= \dfrac{n_0}{N} = 1 - \dfrac{g\zeta(3/2)}{n\lambda^3}$

$$= 1 - \frac{g\zeta(3/2)}{n\lambda_c^3}\left(\frac{\lambda_c}{\lambda}\right)^3$$

$$= 1 - \left(\frac{T}{T_c}\right)^{3/2}$$





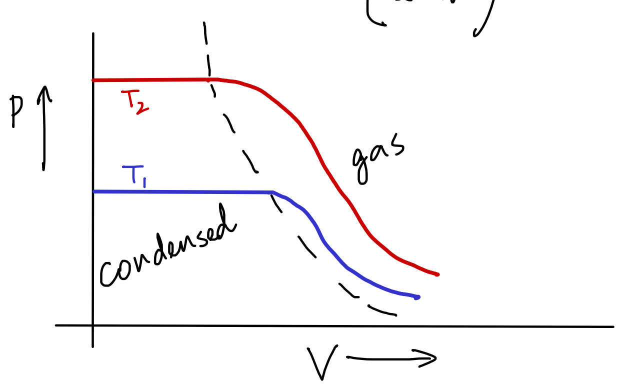
Below T_c a δ-function appears at $\vec{k}=0$.

Equation of state

As z becomes 1, the pressure becomes independent of the density below T_c.

$$\Rightarrow \quad \frac{P\lambda^3}{k_BT} = g_{5/2}(1)\, g = \zeta(5/2)\, g = 1.341\, g$$

$$\Rightarrow \quad P = \zeta(5/2)\, g \left(\frac{m}{2\pi\hbar^2}\right)^{3/2}(k_BT)^{5/2}$$



Volume at the transition line $= V^*$

$$\frac{N}{V^*}\lambda^3 = \zeta(3/2)\, g$$

$$U = \frac{3}{2} PV = \frac{3V}{2} \zeta(5/2)\, g \left(\frac{m}{2\pi\hbar^2}\right)^{3/2} (k_B T)^{5/2}$$

$$\Rightarrow \frac{C_V}{N k_B} = \frac{15}{4} \zeta(5/2)\, g \left(\frac{m}{2\pi\hbar^2}\right)^{3/2} \frac{(k_B T)^{3/2}}{n} \quad \text{---(1)}$$

In the gas phase,

$$P = \frac{k_B T g}{\lambda^3} g_{5/2}(z)$$

$$\Rightarrow C_V = \frac{3}{2} V \left(\frac{\partial P}{\partial T}\right)_V = \frac{3}{2} V g \left[g_{5/2}(z) \frac{d}{dT}\left(\frac{k_B T}{\lambda^3}\right) + \frac{k_B T}{\lambda^3} \frac{\partial}{\partial T} g_{5/2}(z) \right]$$

We know, $z \dfrac{d g_k(z)}{dz} = g_{k-1}(z) \Rightarrow \dfrac{\partial}{\partial T} g_{5/2}(z) = \dfrac{1}{z} g_{3/2}(z) \left(\dfrac{\partial z}{\partial T}\right)_V$

$\because \quad n\lambda^3 = g_{3/2}(z)\, g$

$$\left[\frac{\partial}{\partial T}(n\lambda^3)\right]_V = \frac{g}{z} g_{1/2}(z) \left(\frac{\partial z}{\partial T}\right)_V$$

$$\Rightarrow \left(\frac{\partial z}{\partial T}\right)_V = \frac{nz}{g g_{1/2}(z)} \frac{d\lambda^3}{dT} = \frac{nz}{g g_{1/2}(z)} \frac{d}{dT}\left(\frac{2\pi\hbar^2}{m k_B T}\right)^{3/2} = \frac{nz}{g g_{1/2}(z)} \left(\frac{2\pi\hbar^2}{m k_B}\right)^{3/2} \left(-\frac{3}{2T^{5/2}}\right)$$

$$= -\frac{3nz}{2g g_{1/2}(z)} \frac{\lambda^3}{T}$$

$$\frac{d}{dT}\left(\frac{k_B T}{\lambda^3}\right) = \left(\frac{m}{2\pi\hbar^2}\right)^{3/2} (k_B)^{5/2} \frac{5}{2} T^{3/2} = \frac{5}{2} k_B/\lambda^3$$

$$\Rightarrow C_V = \frac{3Vg}{2}\left[g_{5/2}(z) \frac{5}{2} k_B/\lambda^3 - \frac{k_B T}{\lambda^3 z g} g_{3/2}(z) \frac{3nz}{2g_{1/2}(z)} \frac{\lambda^3}{T} \right]$$

$$\Rightarrow \frac{C_V}{N k_B} = \frac{15}{4} \frac{g_{5/2}(z)}{n\lambda^3} g - \frac{9}{4} \frac{g_{3/2}(z)}{g_{1/2}(z)} = \frac{15}{4} \frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4} \frac{g_{3/2}(z)}{g_{1/2}(z)}$$

High temperature: $\quad g_k(z) \sim z \Rightarrow \dfrac{C_V}{N k_B} \sim \dfrac{3}{2}$

Near T_c, $\quad \dfrac{C_V}{N k_B} = \dfrac{15}{4} \dfrac{g_{5/2}(1)}{g_{3/2}(1)} - \dfrac{9}{4} \dfrac{g_{3/2}(1)}{g_{1/2}(1)}$

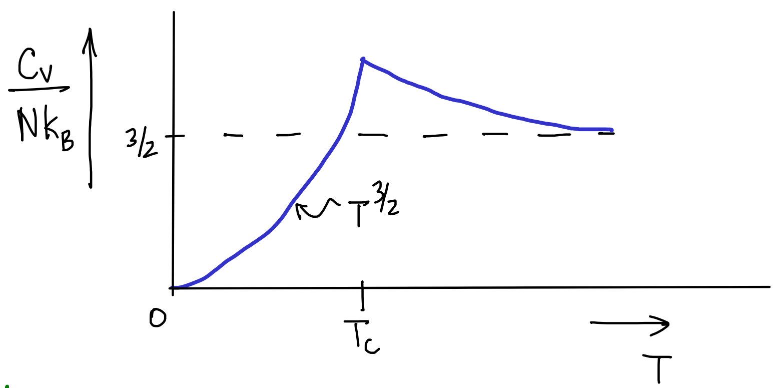
$g_{1/2}(1) = \infty$ $\because$ it is the derivative of $g_{3/2}(z)$ at $z=1$.

$$\therefore \frac{C_V}{Nk_B} = \frac{15}{4}\frac{\zeta(5/2)}{\zeta(3/2)} = 1.93 > \frac{3}{2} \quad \text{---} (2)$$

The value of $\frac{C_V}{Nk_B}$ as T approaches T_c from below is the same (from Eq. 1), since

$$\frac{15}{4}\zeta(5/2)\, g \left(\frac{m}{2\pi\hbar^2}\right)^{3/2} \frac{(k_B T_c)^{3/2}}{n} = \frac{15}{4}\frac{\zeta(5/2)}{n\lambda_c^3} g = \frac{15}{4}\frac{\zeta(5/2)}{\zeta(3/2)}$$

$$\frac{C_V}{Nk_B} = \frac{15}{4}\frac{\zeta(5/2)}{\zeta(3/2)}$$

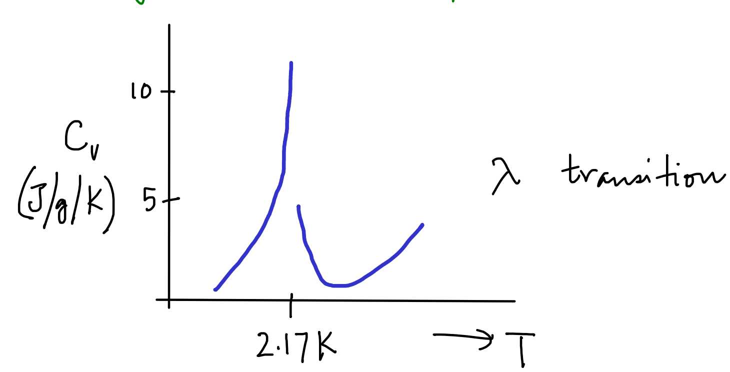


Liquid Helium

The ^4He atom is a boson. It is therefore expected to undergo Bose-Einstein condensation at low temperatures. Because of interactions, liquid ^4He cannot be treated as an ideal Bose gas.

Below T_c, $C_V \propto T^3$

^4He attains a **superfluid** state below T_c
↳ vanishing viscosity



$T_c = 2.17$ K, $n_c = 2.16 \times 10^{22}/\text{cm}^3$

Using the ideal Bose gas expression,

$$T_c = \left[\left(\frac{1}{g\,\zeta(3/2)}\right)^{2/3} \frac{2\pi\hbar^2}{mk_B}\right] n^{2/3} \simeq 3.1\text{ K}$$

(Here m is the mass of a ^4He atom and $g=1$.)