$$\Rightarrow \boxed{\begin{aligned} \frac{P\lambda^3}{k_BT} &= z \mp \frac{z^2}{2^{5/2}} + \frac{z^3}{3^{5/2}} \mp \frac{z^4}{4^{5/2}} + \cdots, \\ \text{and,} \quad n\lambda^3 &= z \mp \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} \mp \frac{z^4}{4^{3/2}} + \cdots \end{aligned}}$$

Equations of state for the ideal Fermi and Bose gases in parametric form.

For the classical ideal gas, z can be eliminated from the above equations and a relation between P, n and k_BT is obtained $[P = nk_BT]$. This is not possible in closed form for the Fermi and Bose gases.

The series in the equations of state are instances of the Fermi and Bose functions.

$$f_k(z) = \sum_{l=1}^{\infty} (-1)^{l+1} \frac{z^l}{l^k} \quad (\text{Fermi}) \quad \text{and} \quad g_k(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^k} \quad (\text{Bose})$$

Integral Representations:

$$\left.\begin{matrix} f_k(z) \\ g_k(z) \end{matrix}\right\} = \frac{2^{2k-1}\,\Gamma(k-\tfrac{1}{2})}{\sqrt{\pi}\;\Gamma(2k-1)} \int_0^{\infty} \frac{x^{2k-1}\,dx}{z^{-1}e^{x^2} \pm 1}$$

Recursion relations:

$$z\frac{d}{dz} f_k(z) = -f_{k-1}(z) \quad \text{and} \quad z\frac{d}{dz} g_k(z) = g_{k-1}(z)$$

The equations of state can be summarized as,

$$\boxed{\frac{P\lambda^3}{k_BT} = \begin{cases} f_{5/2}(z) \\ g_{5/2}(z) \end{cases} \quad \text{and} \quad n\lambda^3 = \begin{cases} f_{3/2}(z) \\ g_{3/2}(z) \end{cases} \quad \text{for the} \begin{cases} \text{Fermi gas} \\ \text{Bose gas} \end{cases}}$$

The Fermi functions have a singularity at $z = -1$ (unphysical region).

The Bose functions have a singularity at $z = 1 \Rightarrow$ Bose-Einstein condensation.

High Temperature / Classical limit

At high temperatures, λ is small $(n\lambda^3 \ll 1)$. In this limit, we will invert the equations of state and try to eliminate z.

Leading order: $\quad z = n\lambda^3$ and $\dfrac{P\lambda^3}{k_BT} = z \Rightarrow P = nk_BT$ (classical ideal gas)

$$\text{Also } n_{\vec{k}} = \frac{1}{e^{\beta\epsilon_{\vec{k}}}/z \pm 1} \approx n\lambda^3 e^{-\beta\epsilon_{\vec{k}}} \quad \left(\begin{array}{c}\text{Maxwell-}\\\text{Boltzmann}\end{array}\right)$$

Corrections :

$$n\lambda^3 = z \mp \frac{z^2}{2^{3/2}} + \cdots \Rightarrow z = n\lambda^3 \pm \frac{n^2\lambda^6}{2^{3/2}} + \cdots$$

$$\Rightarrow e^{\beta\mu} = n\lambda^3\left(1 \pm \frac{n\lambda^3}{2^{3/2}} + \cdots\right)$$

$$\Rightarrow \mu = k_BT\left[\ln(n\lambda^3) + \ln\left(1 \pm \frac{n\lambda^3}{2^{3/2}} + \cdots\right)\right]$$

$$= k_BT\left[\ln(n\lambda^3) \pm \underbrace{\frac{n\lambda^3}{2^{3/2}}}_{\text{quantum correction}}\right]$$

$\Rightarrow$ The quantum correction to the chemical potential of an ideal gas has different signs for bosons and fermions. We will see below that the same is true for the pressure.

$$\frac{P\lambda^3}{k_BT} = z \mp \frac{z^2}{2^{5/2}} + \cdots$$

$$= n\lambda^3\left(1 \pm \frac{n\lambda^3}{2^{3/2}} + \cdots\right) \mp \frac{1}{2^{5/2}}(n\lambda^3)^2\left[\left(1 \pm \frac{n\lambda^3}{2^{3/2}} + \cdots\right)\right]^2 + \cdots$$

$$= n\lambda^3 + \left(\pm\frac{1}{2^{3/2}} \mp \frac{1}{2^{5/2}}\right)(n\lambda^3)^2 + \cdots$$

$$= n\lambda^3 \pm \frac{2-1}{2^{5/2}}(n\lambda^3)^2 + \cdots$$

$$= n\lambda^3\left[1 \pm \frac{1}{2^{5/2}}(n\lambda^3)^2 + \cdots\right]$$

$$\Rightarrow P = nk_BT\left[1 \pm \frac{1}{2^{5/2}}(n\lambda^3)^2 + \cdots\right]$$

$\Rightarrow$ At the same temperature and density,

$$P_{\text{Bose gas}} < P_{\text{Classical gas}} < P_{\text{Fermi gas}}$$

$\Rightarrow$ Fermions feel effective repulsion and bosons feel effective attraction due to symmetry requirements.

The Fermi Gas

At absolute zero, the Fermi-Dirac distribution function becomes a step function.

$$n_{\vec{k}} = \frac{1}{e^{\beta(\epsilon_{\vec{k}}-\mu)}+1} \xrightarrow[T\to 0]{} \Theta(\mu-\epsilon_{\vec{k}}) = \begin{cases} 1 & \text{if } \epsilon_{\vec{k}}<\mu \\ 0 & \text{if } \epsilon_{\vec{k}}>\mu \end{cases}$$

The **Fermi energy** ϵ_F is the value of the chemical potential at absolute zero, for a given density.

$$\epsilon_F(n) = \mu(n,0) \quad \text{where } \mu=\mu(n,T)\ , n = \text{number density}$$

$\Rightarrow$ At $T=0$, all states below the Fermi energy are occupied and all states above are empty — Quantum degeneracy.

$$N = \sum_{\substack{\vec{k} \\ \epsilon_{\vec{k}}<\epsilon_F}} 1$$

If the spin of the fermion is S, each momentum state is $(2S+1)$-fold degenerate.

$$\Rightarrow N = \sum_{\substack{\vec{k} \\ \epsilon_{\vec{k}}<\epsilon_F}} (2S+1) = \frac{(2S+1)V}{(2\pi)^3}\int_0^{k_F} d^3k = \frac{(2S+1)V}{(2\pi)^3}\frac{4}{3}\pi k_F^3$$

$$\Rightarrow n = \frac{(2S+1)}{6\pi^2}k_F^3 \quad , \text{ where } k_F \text{ is the Fermi wave vector.}$$

$$\Rightarrow k_F = \left(\frac{6\pi^2 n}{2S+1}\right)^{1/3}$$

The **Fermi momentum**, $p_F = \hbar k_F$.

The Fermi energy, $\epsilon_F = \dfrac{\hbar^2 k_F^2}{2m} = \dfrac{\hbar^2}{2m}\left(\dfrac{6\pi^2 n}{2S+1}\right)^{2/3}$

The ground state ($T=0$) internal energy is,

$$U_0 = \sum_{k<k_F} (2S+1)\, \epsilon_{\vec{k}} = \frac{(2S+1)V}{(2\pi)^3} \int_0^{k_F} 4\pi k^2 dk \frac{\hbar^2 k^2}{2m} = \frac{(2S+1)}{20\pi^2} V \hbar^2 k_F^5 .$$

We had seen earlier that $U = \frac{3}{2} PV$.

$\Rightarrow$ At absolute zero, the pressure of a Fermi gas is,

$$P_F = \frac{2}{3}\frac{U_0}{V} = \frac{2}{3}\frac{(2S+1)}{20\pi^2}\hbar^2 k_F^5 = \frac{2}{3}\frac{(2S+1)}{20\pi^2}\left(\frac{\hbar^2 k_F^2}{2m}\right) 2m\, k_F^3$$

$$= \frac{(2S+1)}{15\pi^2}\epsilon_F \frac{6\pi^2 n}{2S+1} = \frac{2}{5} n \epsilon_F$$

Note: A classical gas has zero pressure at absolute zero. For a Fermi gas, since the zero momentum state cannot hold more than $2S+1$ particles, there are particles with non-zero momentum even at $T=0$, which contribute towards the pressure of the Fermi gas.

Metal / Electron gas:

$S = 1/2$, $n \approx 10^{22}/cm^3 = 10^{28}/m^3$

$$\Rightarrow \epsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$$

$\hbar = 197 \text{ MeV fm}/c$

$m = 0.51 \text{ MeV}/c^2$

$$= \frac{(200\times10^6\times10^{-15})^2 \times (3\pi^2\times10^{28})^{2/3}}{2\times0.5\times10^6} eV$$

$(3\pi^2)^{2/3} \approx 3^2 \approx 10$

$10^{2/3} = 100^{1/3} \approx 5$

$$\approx \frac{4\times10^{-14}\times10\times5\times10^{18}}{10^6} eV = 2\,eV$$

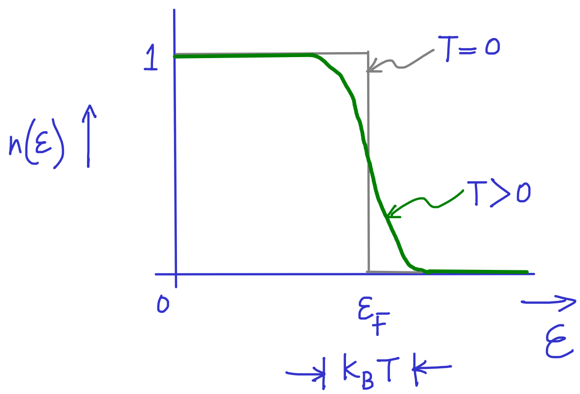
$$\Rightarrow \epsilon_F \approx 2eV$$

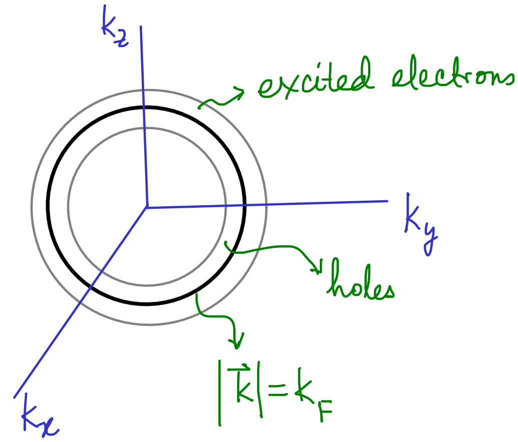
$$\Rightarrow P_F = \frac{2}{5} n\epsilon_F \approx \frac{2}{5}\times10^{28}\times 2\times1.6\times10^{-19} \text{ Pa} \sim 10^9 \text{ Pa} \approx 10^4 \text{ atm}$$

The **Fermi temperature** is defined using the relation, $k_B T_F = \varepsilon_F$.

For the electron gas, $T_F \approx \dfrac{2 \times 1.6 \times 10^{-19}}{1.38 \times 10^{-23}} \approx 2 \times 10^4\,K$.

At low but non-zero temperatures ($T \ll T_F$) some electrons below the Fermi level get excited and acquire energies that are higher than ε_F. This creates "holes" below the Fermi level.





Fraction of excited electrons $\sim \dfrac{4\pi k_F^2\,\delta k}{\frac{4}{3}\pi k_F^3} \sim \dfrac{3\,\delta k}{k} \sim \dfrac{3}{2}\dfrac{\delta\varepsilon}{\varepsilon_F} \sim \dfrac{\delta\varepsilon}{\varepsilon_F} = \dfrac{T}{T_F}$ $\quad \because \varepsilon \propto k^2$

At room temperature, fraction of excited electrons $\sim \dfrac{300}{20000} = 1.5\%$

Since the excitation energy per particle is of the order of $k_B T$, the total internal energy is

$$U \sim N \frac{T}{T_F} k_B T = \frac{N k_B T^2}{T_F}$$

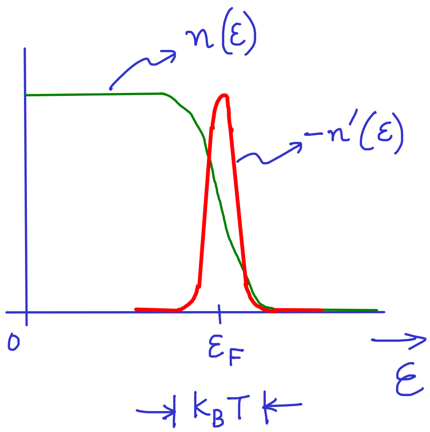
$\Rightarrow$ Specific heat capacity, $C = \dfrac{dU}{dT} \sim \dfrac{N k_B T}{T_F}$

$\Rightarrow$ Electronic contribution to the specific heat can be neglected at room temperatures. [$\because T \ll T_F$]

We can get a more exact form for the specific heat capacity using a low temperature expansion of the Fermi functions.

$$f_k(z) = \frac{2^{2k-1}\,\Gamma(k-\frac{1}{2})}{\sqrt{\pi}\;\Gamma(2k-1)} \int_0^\infty \frac{x^{2k-1}\,dx}{z^{-1}e^{x^2}+1}, \qquad z = e^{\beta\mu}$$

When $T \ll T_F$, the derivative of the Fermi distribution formula is sharply peaked near $\varepsilon = \varepsilon_F$. Therefore, if we integrate $f_k(z)$ by parts, we may get a δ-like function.



$$\frac{d}{dx}\left(\frac{1}{e^{x^2-\beta\mu}+1}\right) \to 0 \quad \text{if} \quad |x^2-\beta\mu| \gg 1$$

Let $x^2 = \beta\mu + t \;\Rightarrow\; 2x\,dx = dt$

$$\int_0^\infty \frac{x^{2k-1}\,dx}{z^{-1}e^{x^2}+1} = \frac{1}{2}\int_{-\beta\mu}^{\infty} \frac{(t+\beta\mu)^{k-1}}{e^t+1}\,dt = \left.\frac{(t+\beta\mu)^k}{2k(e^t+1)}\right|_{-\beta\mu}^{\infty} + \frac{1}{2k}\int_{-\beta\mu}^{\infty}(t+\beta\mu)^k \frac{e^t}{(e^t+1)^2}\,dt$$

$$= \frac{(\beta\mu)^k}{2k}\int_{-\beta\mu}^{\infty}\frac{e^t}{(e^t+1)^2}\left[1+\frac{kt}{\beta\mu}+\frac{k(k-1)}{2}\left(\frac{t}{\beta\mu}\right)^2+\cdots\right]dt$$

Since the integrand is sharply peaked near $t=0$, we can replace the lower limit with $-\infty$.

$$\Rightarrow \int_0^\infty \frac{x^{2k-1}\,dx}{z^{-1}e^{x^2}+1} \approx \frac{(\beta\mu)^k}{2k}\sum_{r=0}^{\infty}\binom{k}{r}\left(\frac{1}{\beta\mu}\right)^r \int_{-\infty}^{\infty}\frac{t^r e^t\,dt}{(e^t+1)^2}$$

$\dfrac{e^t}{(e^t+1)^2}$ is an even function

$$\binom{k}{r} = \frac{k(k-1)\cdots(k-r+1)}{r!}$$

$$\Rightarrow \int_0^\infty \frac{x^{2k-1}\,dx}{z^{-1}e^{x^2}+1} \approx \frac{(\beta\mu)^k}{k}\sum_{r\in\text{even}}\binom{k}{r}\left(\frac{1}{\beta\mu}\right)^r \int_0^\infty\frac{t^r e^t\,dt}{(e^t+1)^2}$$

$$\int_0^\infty \frac{e^t\,dt}{(e^t+1)^2} = \int_2^\infty \frac{dz}{z^2} = -\frac{1}{z}\Big|_2^\infty = \frac{1}{2}$$

$e^t+1=z$
$e^t dt = dz$

$$\int_0^\infty \frac{t^2 e^t\,dt}{(e^t+1)^2} = -\lim_{\alpha\to1}\frac{d}{d\alpha}\int_0^\infty \frac{t\,dt}{e^{\alpha t}+1} = -\lim_{\alpha\to1}\frac{d}{d\alpha}\frac{1}{\alpha^2}\int_0^\infty \frac{x\,dx}{e^x+1} = 2\int_0^\infty \frac{x\,dx}{e^x+1}$$

Riemann Zeta Function

$$\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z} = \frac{1}{\Gamma(z)}\int_0^\infty \frac{x^{z-1}\,dx}{e^x-1} = \frac{1}{(1-2^{1-z})\Gamma(z)}\int_0^\infty \frac{x^{z-1}\,dx}{e^x+1}$$

z	-1	0	1	2	4
$\zeta(z)$	$-1/12$	$-1/2$	∞	$\pi^2/6$	$\pi^4/90$

$$\int_0^\infty \frac{x\,dx}{e^x+1} = \Gamma(2)\zeta(2)(1-2^{1-2}) = \frac{\pi^2}{12}$$

$$\int_0^\infty \frac{x^{2k-1}\,dx}{z^{-1}e^{x^2}+1} \approx \frac{(\beta\mu)^k}{k}\left[\frac{1}{2} + \frac{k(k-1)}{2}\,\frac{\pi^2}{6}\left(\frac{1}{\beta\mu}\right)^2 + \cdots\right] \quad \text{where } \beta\mu = \ln z \gg 1$$

$$\Rightarrow f_k(z) \approx \frac{2^{2k-1}\,\Gamma(k-\frac{1}{2})}{\sqrt{\pi}\;\Gamma(2k-1)}\,\frac{(\beta\mu)^k}{2k}\left[1 + k(k-1)\frac{\pi^2}{6}\left(\frac{1}{\beta\mu}\right)^2 + \cdots\right]$$

$$\Rightarrow f_{3/2}(z) \approx \frac{4}{\sqrt{\pi}}\,\frac{2}{6}(\beta\mu)^{3/2}\left[1 + \frac{3}{4}\,\frac{\pi^2}{6}\left(\frac{1}{\beta\mu}\right)^2 + \cdots\right]$$

$$\approx \frac{4}{3\sqrt{\pi}}\left[(\ln z)^{3/2} + \frac{\pi^2}{8}(\ln z)^{-1/2} + \cdots\right]$$

and
$$f_{5/2}(z) \approx \frac{16}{\sqrt{\pi}\times 6}\,\frac{2}{5}\,\frac{1}{2}(\beta\mu)^{5/2}\left[1 + \frac{15}{4}\,\frac{\pi^2}{6}\left(\frac{1}{\beta\mu}\right)^2 + \cdots\right]$$

$$\approx \frac{8}{15\sqrt{\pi}}\left[(\ln z)^{5/2} + \frac{5\pi^2}{8}(\ln z)^{1/2} + \cdots\right]$$

$$\Rightarrow n\lambda^3 \approx \frac{4g}{3\sqrt{\pi}}\left[(\ln z)^{3/2} + \frac{\pi^2}{8}(\ln z)^{-1/2} + \cdots\right] \quad ---(1)$$

$g = (2S+1)$ is the spin degeneracy
$g = 2$ for electrons

and $$\frac{P\lambda^3}{k_B T} \approx \frac{8g}{15\sqrt{\pi}}\left[(\ln z)^{5/2} + \frac{5\pi^2}{8}(\ln z)^{1/2} + \cdots\right] \quad ---(2)$$

Keeping the first term in Eq. (1), $\ln z \sim \left(\frac{3n\sqrt{\pi}\,\lambda^3}{4g}\right)^{2/3}$

$$\Rightarrow \ln z \approx \left[\frac{3n\sqrt{\pi}}{4g}\right]^{2/3}\lambda^2$$

$$\epsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3}$$

$$= \left[\frac{3n\sqrt{\pi}}{4g}\right]^{2/3}\frac{2\pi\hbar^2}{mk_BT} = \left(\frac{6\pi^2 n}{g}\right)^{2/3}\frac{2}{4^{2/3}\times 2^{2/3}}\,\frac{\hbar^2}{mk_BT} = \frac{\epsilon_F}{k_BT}$$

$$= \frac{T_F}{T}$$