Many particle wavefunctions of bosons and fermions.

Suppose we have N particles occupying single particle states $\{s_1, s_2, \dots, s_N\}$. Their joint wavefunction can be denoted by,

$$\psi_{s_1 s_2 \dots s_N}(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N),$$

where the i^{th} particle is in state s_i and at location $\vec{r}_i$.

For indistinguishable quantum particles, the quantum state must not change on interchanging two particles, i.e., the wavefunction can change at most by a phase.

$$\therefore \psi_{\dots s_j \dots s_i \dots}(\dots, \vec{r}_j, \dots, \vec{r}_i, \dots) = e^{i\theta}\, \psi_{\dots s_i \dots s_j \dots}(\dots, \vec{r}_i, \dots, \vec{r}_j, \dots)$$

If we do another exchange of the same particles, then we must get back the same wavefunction.

$$\therefore e^{2i\theta} = 1 \Rightarrow e^{i\theta} = \pm 1$$

These correspond to bosons and fermions.

For bosons, the joint wavefunction is symmetric with respect to the exchange of two bosons, i.e.,

$$\psi_{\dots s_j \dots s_i \dots}(\dots, \vec{r}_j, \dots, \vec{r}_i, \dots) = \psi_{\dots s_i \dots s_j \dots}(\dots, \vec{r}_i, \dots, \vec{r}_j, \dots)$$

For fermions, the joint wavefunction is antisymmetric with respect to the exchange of two bosons, i.e.,

$$\psi_{\dots s_j \dots s_i \dots}(\dots, \vec{r}_j, \dots, \vec{r}_i, \dots) = -\psi_{\dots s_i \dots s_j \dots}(\dots, \vec{r}_i, \dots, \vec{r}_j, \dots)$$

This implies Pauli's exclusion principle, i.e. two fermions cannot be in the same state.

$$\psi_{\ldots s_i \ldots s_i \ldots}(\ldots,\vec{r}_i,\ldots,\vec{r}_i,\ldots) = -\psi_{\ldots s_i \ldots s_i \ldots}(\ldots,\vec{r}_i,\ldots,\vec{r}_i,\ldots)$$

$$\Rightarrow 2\psi_{\ldots s_i \ldots s_i \ldots}(\ldots,\vec{r}_i,\ldots,\vec{r}_i,\ldots) = 0$$

$$\Rightarrow \psi_{\ldots s_i \ldots s_i \ldots}(\ldots,\vec{r}_i,\ldots,\vec{r}_i,\ldots) = 0 \Rightarrow \text{Zero probability (impossible)}$$

It is often convinient to write the joint wavefunction of many particles as a linear combination of products of single particle wavefunctions, i.e.

$$\psi(\vec{r}_1,\vec{r}_2\cdots,r_N) = \sum_{\alpha_1\alpha_2\ldots} C_{\alpha_1\alpha_2\cdots\alpha_N}\,\varphi_{\alpha_1}(\vec{r}_1)\,\varphi_{\alpha_2}(\vec{r}_2)\cdots\varphi_{\alpha_N}(\vec{r}_N)$$

Such linear combinations must respect the exchange symmetry described above.

For fermions, this is achieved by the following form.

$$\psi(\vec{r}_1,\vec{r}_2\cdots,r_N) = \frac{1}{\sqrt{N!}}\begin{vmatrix} \varphi_1(\vec{r}_1) & \varphi_1(\vec{r}_2) & \cdots & \varphi_1(\vec{r}_N) \\ \varphi_2(\vec{r}_1) & \varphi_2(\vec{r}_2) & \cdots & \varphi_2(\vec{r}_N) \\ \vdots & \vdots & & \vdots \\ \varphi_N(\vec{r}_1) & \varphi_N(\vec{r}_2) & \cdots & \varphi_N(\vec{r}_N) \end{vmatrix}$$

where the N particles occupy single particle states $\{\varphi_1,\varphi_2,\ldots\varphi_N\}$.

The above form is known as a Slater determinant.

Expanding the determinant, we get,

$$\psi(\vec{r}_1,\vec{r}_2\cdots,r_N) = \frac{1}{\sqrt{N!}}\sum_P (-1)^{\mathcal{P}(P)}\,\varphi_{P_1}(\vec{r}_1)\,\varphi_{P_2}(\vec{r}_2)\cdots\varphi_{P_N}(\vec{r}_N),$$

where $P \equiv (P_1 P_2\cdots P_N)$ is a permutation of $(1\,2\,3\ldots N)$ and $\mathcal{P}(P)$ is its parity, i.e. number of pairwise exchanges required to go from $(1\,2\,3\ldots N)$ to P.

For bosons,

$$\psi(\vec{r}_1, \vec{r}_2, \dots, r_N) = \frac{1}{\sqrt{N!}} \sum_P \varphi_{p_1}(\vec{r}_1)\, \varphi_{p_2}(\vec{r}_2) \cdots \varphi_{p_N}(\vec{r}_N),$$

which is called a permanent.

For completeness, we mention the **spin-statistics theorem** here. Particles with integer spins (in units of $\hbar$) obey Bose-Einstein statistics and those with half integer spins (odd multiple of $\frac{1}{2}$) obey Fermi-Dirac statistics.

The thermal deBroglie wavelength

The atoms/molecules in a gas are wave packets that are sharply localized, and therefore appear classical at high temperatures. As the temperature is lowered, the wave packets spread out and the extent of delocalization is quantified by the de Broglie wavelength h/p.

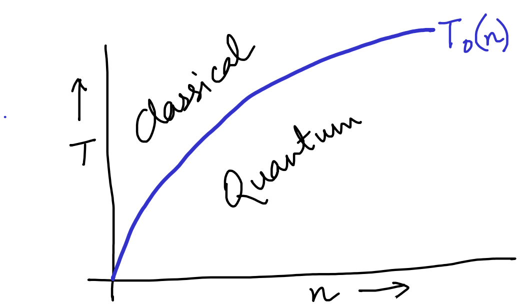
The thermal de Broglie wavelength is defined as,

$$\lambda = \sqrt{\frac{2\pi\hbar^2}{m k_B T}},$$

where the dimensionless factors are chosen for convenience.

$n\lambda^3 \ll 1$ describes the classical regime, where n = number density.

$n\lambda^3 = 1$ corresponds to the "quantum degeneracy temperature", T_0, below which the classical description breaks down.



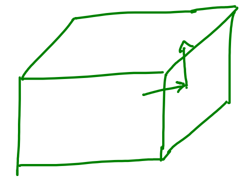
$$n\left(\frac{2\pi\hbar^2}{m k_B T_0}\right)^{3/2} = 1$$

$$\Rightarrow T_0 = \left(\frac{2\pi\hbar^2}{m k_B}\right) n^{2/3}$$

	n (in cm^{-3})	T_0 (in K)
H_2 gas	2×10^{19}	0.05
^4He liquid	2×10^{22}	2
Electrons in a metal	10^{22}	10^4

Equations of state of ideal Bose and Fermi gases

To get the equation of state, we need to relate the pressure to the density at a given temperature.



Pressure : From the kinetic theory,

$$P = \sum_{\substack{\vec{k} \\ v_x > 0}} \frac{2mv_x \, v_x}{V} n_{\vec{k}}$$

$$= \sum_{k} \frac{mv_x^2}{V} n_{\vec{k}}$$

$$P = \frac{\text{Force}}{\text{Area}} = \frac{\Delta p}{(\text{Area}) \Delta t}$$

$$= \frac{2mv_x}{V} \frac{\Delta x}{\Delta t}$$

$$= \frac{2mv_x^2}{V}$$

$$\epsilon_{\vec{k}} = \frac{1}{2}mv^2 = \frac{3}{2}mv_x^2$$

$$\Rightarrow \boxed{P = \frac{2}{3V} \sum_{\vec{k}} \epsilon_{\vec{k}} n_{\vec{k}} = \frac{2U}{3V}}$$, where U is the internal energy of the gas.

$$\Rightarrow P = \frac{2}{3} \int \frac{d^3k}{(2\pi)^3} \frac{\epsilon_{\vec{k}}}{e^{\beta(\epsilon_{\vec{k}} - \mu)} \pm 1}$$

$+$: Fermi gas
$-$: Bose gas

$$= \frac{2}{3(2\pi)^3} \int_0^\infty 4\pi k^2 dk \frac{\hbar^2 k^2 / 2m}{\frac{1}{z} e^{\beta \hbar^2 k^2 / 2m} \pm 1}$$

$$z = \text{fugacity} = e^{\beta\mu}$$

$$\beta \frac{\hbar^2 k^2}{2m} = x^2$$

$$= \frac{\hbar^2}{6\pi^2 m} \left(\frac{2mk_BT}{\hbar^2} \right)^{5/2} \int_0^\infty \frac{x^4 dx}{\frac{1}{z} e^{x^2} \pm 1}$$

$$= \frac{2^{5/2} k_BT}{6\pi^2} (2\pi)^{3/2} \left(\frac{mk_BT}{2\pi\hbar^2} \right)^{3/2} \int_0^\infty \frac{x^4 dx}{\frac{1}{z} e^{x^2} \pm 1}$$

$$\Rightarrow \boxed{P = \frac{8 k_BT}{3\sqrt{\pi} \lambda^3} \int_0^\infty \frac{x^4 dx}{\frac{1}{z} e^{x^2} \pm 1}}$$

Density : We know,

$$N = \sum_{\vec{k}} n_{\vec{k}} = \frac{V}{(2\pi)^3} \int d^3k \, n_{\vec{k}}$$

$$\Rightarrow n = \frac{1}{(2\pi)^3} \int d^3k \, n_{\vec{k}}$$

$$\Rightarrow n = \frac{1}{(2\pi)^3}\int_0^\infty \frac{4\pi k^2 dk}{\frac{1}{z}e^{\beta\hbar^2k^2/2m} \pm 1}$$

$$= \frac{4\pi}{(2\pi)^3}\left(\frac{2mk_BT}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{x^2 dx}{\frac{1}{z}e^{x^2} \pm 1}, \quad \text{where, as before, } x^2 = \beta\frac{\hbar^2k^2}{2m}$$

$$= \frac{4\pi \times 2^{3/2}}{(2\pi)^{3/2}}\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}\int_0^\infty \frac{x^2 dx}{\frac{1}{z}e^{x^2} \pm 1}$$

$$\Rightarrow \boxed{n = \frac{4}{\lambda^3\sqrt{\pi}}\int_0^\infty \frac{x^2 dx}{\frac{1}{z}e^{x^2} \pm 1}}$$

Now we will examine the integrals in the pressure and density expressions more carefully.

$$\int_0^\infty \frac{x^2 dx}{\frac{1}{z}e^{x^2} \pm 1} = z\int_0^\infty \frac{x^2 e^{-x^2} dx}{1 \pm z e^{-x^2}} = z\int_0^\infty x^2 e^{-x^2}\left(1 \mp z e^{-x^2} + z^2 e^{-2x^2} \mp z^3 e^{-3x^2} + \dots\right)dx$$

The various terms are of the form $z^n \int_0^\infty x^2 e^{-mx^2} dx$.

We know, $\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} x^2 e^{-x^2/2\sigma^2} dx = \langle x^2 \rangle = \sigma^2$

$$\Rightarrow \int_0^\infty x^2 e^{-x^2/2\sigma^2} dx = \frac{1}{2}\sigma^3\sqrt{2\pi}$$

$$\frac{1}{2\sigma^2} = m \quad \Rightarrow \sigma^2 = \frac{1}{2m}$$

$$\Rightarrow \int_0^\infty x^2 e^{-mx^2} dx = \frac{\sqrt{2\pi}}{2}\left(\frac{1}{2m}\right)^{3/2} = \frac{\sqrt{\pi}}{4} m^{-3/2}$$

$$\Rightarrow \int_0^\infty \frac{x^2 dx}{\frac{1}{z}e^{x^2} \pm 1} = \frac{\sqrt{\pi}}{4}\left[z \mp \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} \mp \frac{z^4}{4^{3/2}} + \dots\right]$$

Similarly,

$$\int_0^\infty \frac{x^4\,dx}{\frac{1}{z}e^{x^2}\pm 1} = z\int_0^\infty \frac{x^4 e^{-x^2}\,dx}{1\pm z e^{-x^2}} = z\int_0^\infty x^4 e^{-x^2}\left(1 \mp z e^{-x^2} + z^2 e^{-2x^2} \mp z^3 e^{-3x^2} + \dots\right)dx$$

The various terms are of the form $z^n \int_0^\infty x^4 e^{-mx^2}\,dx$.

We know, $\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} x^4 e^{-x^2/2\sigma^2}\,dx = 3\sigma^4$ [4th moment of a Gaussian is thrice the square of its variance.]

$$\Rightarrow \int_0^\infty x^4 e^{-x^2/2\sigma^2}\,dx = \frac{3}{2}\sigma^5\sqrt{2\pi}$$

$$\frac{1}{2\sigma^2} = m \quad \Rightarrow \sigma^2 = \frac{1}{2m}$$

$$\Rightarrow \int_0^\infty x^4 e^{-mx^2}\,dx = \frac{3\sqrt{2\pi}}{2}\left(\frac{1}{2m}\right)^{5/2} = \frac{3\sqrt{\pi}}{8}\, m^{-5/2}$$

$$\Rightarrow \int_0^\infty \frac{x^4\,dx}{\frac{1}{z}e^{x^2}\pm 1} = \frac{3\sqrt{\pi}}{8}\left[z \mp \frac{z^2}{2^{5/2}} + \frac{z^3}{3^{5/2}} \mp \frac{z^4}{4^{5/2}} + \dots\right]$$

$$\Rightarrow \boxed{\begin{aligned} \frac{p\lambda^3}{k_B T} &= z \mp \frac{z^2}{2^{5/2}} + \frac{z^3}{3^{5/2}} \mp \frac{z^4}{4^{5/2}} + \dots, \\ \text{and,}\quad n\lambda^3 &= z \mp \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} \mp \frac{z^4}{4^{3/2}} + \dots \end{aligned}}$$

Equations of state for the ideal Fermi and Bose gases in parametric form.