* Complex Analysis *

- Euler's formula, De Moivre's theorem, Roots of complex numbers, Functions of complex variables
- Analyticity & Cauchy-Riemann conditions
- Singular functions:
  - Poles
  - Branch points
  - Order of singularity
  - Branch cuts
- Integration of a function of a complex variable
- Cauchy's inequality
- Cauchy's integral formula
- Simply and multiply connected regions
- Laurent and Taylor series expansions
- Residues & Residue theorem
- Application in solving definite integrals

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* Definition → A complex number z is an ordered pair of real numbers : $z \equiv (x, y)$ with addition and multiplication defined as follows.

For $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$

$$z_1 + z_2 \equiv (x_1 + x_2, y_1 + y_2)$$

$$z_1 z_2 \equiv (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$$

Numbers of the form $(x, 0)$ are "called" real numbers and those of the form $(0, y)$ are called imaginary numbers.

$$(x, 0) + (0, y) = (x, y) = (x, 0) + (0, 1)(y, 0)$$

$\text{Re } z = x, \quad \text{Im } z = y$

$z_1 = z_2$ iff $x_1 = x_2$ and $y_1 = y_2$

$(0,1) \equiv i \quad , \ z = x + iy \quad$ where $x \equiv (x, 0)$

$i^2 = (0,1) \times (0,1) = -1$

→ Closure under addition & multiplication
→ Additive & multiplicative identity $(0 \,\&\, 1)$
→ Additive inverse $\forall z \quad (-z)$
→ Multiplicative inverse $\forall z$ except 0. $\quad (1/z)$
→ Closure under exponentiation (defined later) $z_1^{z_2}$

Ex: Find the real & imaginary parts of $1/z$

Geometric Interpretation | Modulus

$|z| = \sqrt{x^2 + y^2}$ (Distance from origin)

Ex:

$|z_1 - z_2| = ?$ (Distance b/w z_1 & z_2)

Ex:

Equation of a circle ?

$|z - z_0| = R$

Region inside a circle:

$|z - z_0| \leq R$

Complex conjugate : $\bar{z} = x - iy$

$z\bar{z} = |z|^2$

$z + \bar{z} = 2 \,\text{Re } z$

$z - \bar{z} = 2i \,\text{Im } z$

7/1 **Further properties of moduli**

$$|\bar{z}| = |z|$$

$$|z_1 z_2| = |z_1||z_2|$$

$$|z|^2 = (\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2$$

$$|z| \geq |\mathrm{Re}\, z| \geq \mathrm{Re}\, z \;, \quad |z| \geq |\mathrm{Im}\, z| \geq \mathrm{Im}\, z$$

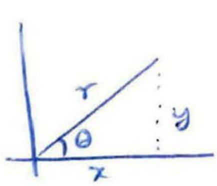
$$|z_1 + z_2| \leq |z_1| + |z_2| \qquad \text{(Triangle Inequality)}$$

Proof:
$$\begin{aligned} |z_1+z_2|^2 &= (z_1+z_2)\overline{(z_1+z_2)} \\ &= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 \\ &= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}\, z_1\bar{z}_2 \\ &\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \\ &= (|z_1| + |z_2|)^2 \end{aligned}$$

$$\Rightarrow |z_1 + z_2| \leq |z_1| + |z_2| \quad \because |\cdot| \geq 0$$

Polar form ~~(Euler's identity formula)~~

$(r, \theta) \rightarrow$ Polar coordinates corresponding to (x, y)



$$r = |z|$$

$$\theta = \tan^{-1}\frac{y}{x} = \arg z$$

Principal value of $\arg z \equiv \mathrm{Arg}\, z \in (-\pi, \pi]$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

Exponential Form / Euler's formula

From Taylor series expansion of cos & sin

$$\cos\theta + i\sin\theta = 1 - \frac{\theta^2}{2} + \cdots + i\left(\theta - \frac{\theta^3}{3!}\cdots\right)$$
$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \cdots$$
$$= e^{i\theta}$$

$$e^{i\theta} = \cos\theta + i\sin\theta \equiv \text{cis}\,\theta$$

$$\cancel{e^{i\theta_1+\theta_2}} \quad e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

$$e^{i2n\pi} = 1 \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$\boxed{z_1 = z_2 \Rightarrow r_1 = r_2 \text{ and } \theta_1 = \theta_2 + 2k\pi,\ k \in \mathbb{Z}}$$

Parametric ~~rep~~ representation of a circle

$$z = z_0 + Re^{i\theta} \qquad 0 \leq \theta < 2\pi$$

$$\left(e^{i\theta}\right)^n = e^{in\theta}$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \text{de Moivre's formula}$$

n^{th} roots of unity

$$z^n = 1$$

$$\left(re^{i\theta}\right)^n = 1 \Rightarrow r^n e^{in\theta} = 1e^{i0}$$

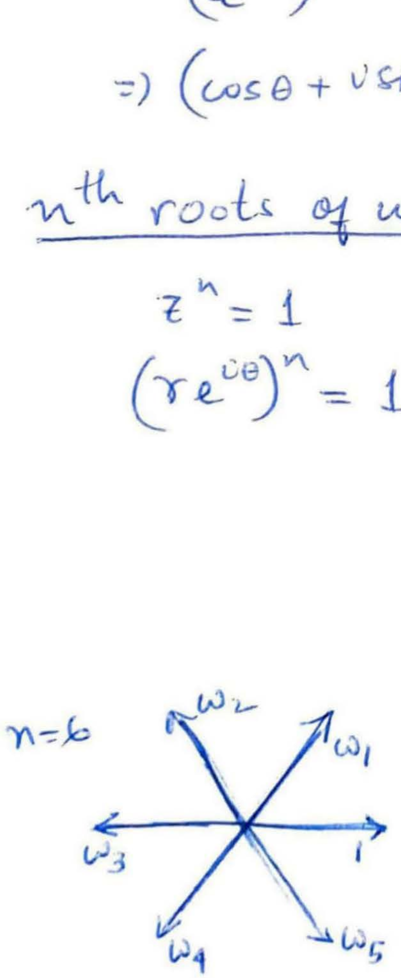
$$\Rightarrow r^n = 1 \Rightarrow r = 1$$

$$n\theta = 2k\pi \Rightarrow \theta = \frac{2k\pi}{n}$$

$\Rightarrow$ n distinct roots:

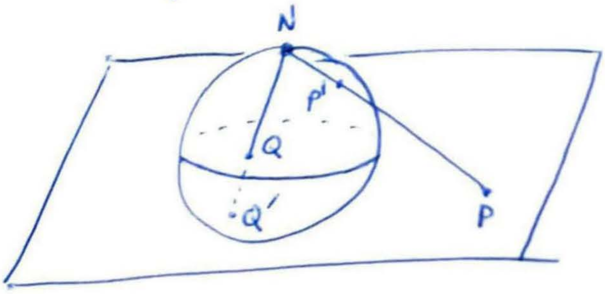
$$z = \exp\left(i\,\frac{2k\pi}{n}\right) \qquad k = 0, 1, 2, \ldots, (n-1)$$

ω_i



Stereographic projection

Complex plane → Surface of sphere of unit radius



North pole → Point at infinity

South pole → origin

~~Lecs 1+2~~ 25/01 Lec Hrs - 2

Ex:

① Show $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i} = -\dfrac{2}{5}$

② $(1-i)^4 = -4$

③ $\left|(2\bar{z}+5)(\sqrt{2}-i)\right| = \sqrt{3}\,|(2z+5)$

④ Use de Moivre's formula to derive

ⓐ $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^3\theta$

ⓑ $\sin 3\theta = 3\cos^2\theta - \sin^3\theta$

⑤ Establish $1+z+z^2+\cdots+z^n = \dfrac{1-z^{n+1}}{1-z}$

From this derive Lagrange's trigonometric identity

$$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta = \frac{1}{2} + \frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right]}{2\sin(\theta/2)}$$

⑥ Sketch the regions

ⓐ $|z-2+i| \leq 1$

ⓑ $|z-4| \geq |z|$

Function of a complex variable

$$f(z) = u(x, y) + i\, v(x, y) = w$$

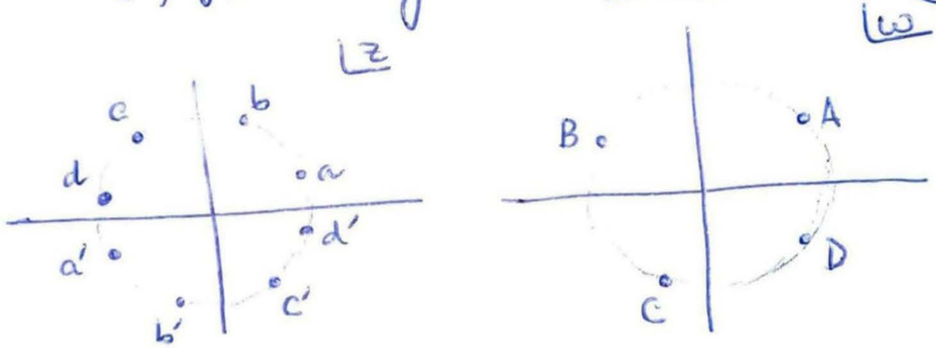
Graphical Representation
1. Plot surfaces $u(x, y)$ & $v(x, y)$ on x, y plane
2. Plot u, v on w plane

Function — mapping from z-plane to w-plane

LIMIT: "$z \to z_0$"
Infinite sequence of points approaching z_0 along any path in z-plane

Continuity at z_0:
$f(z_0)$ exists and $\lim_{z \to z_0} f(z) = f(z_0)$

E.g.
$$f(z) = z^2 = (x + iy)^2$$
$$= x^2 - y^2 + 2ixy$$
$$u(x, y) = x^2 - y^2$$
$$v(x, y) = 2xy$$



Upper half z-plane maps onto the entire w-plane
↳ same for lower half z-plane

Inverse mapping of $z^2 \to z^{1/2}$ looks multivalued.

Convert to single valued by enlarging the domain.

The enlarged domain is called the Riemann surface.

For the square root function, the Riemann surface consists of 2 copies of the complex plane (Riemann sheets)

On sheet 1, z is taken as $z = re^{i\theta}$ with $0 \le \theta < 2\pi$

On sheet 2, $z = re^{i\theta}$ with $2\pi \le \theta < 4\pi$.

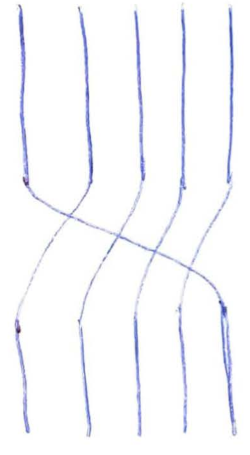
The sheets are connected along an infinitely long line going from $z=0$ to the point at infinity in any direction. On crossing this line, on goes from the first sheet to the second and vice versa.

The points $z=0$ & $z=\infty$ are called branch points and the above line is called a branch cut.
The branch cut can be curved as well. It should be non-intersecting with itself and go from $z=0$ to $z=\infty$.

Thus, the square root function is continuous ~~on the~~ for $0<|z|<\infty$ when the domain is a two-sheeted Riemann surface.

For $f(z)=z^{1/5}$, we need a 5-sheeted Riemann surface

Edge view: across the branch cut



Differentiability:

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$$ Must exist and be unique

~~A function is ANALYTIC at z if it has a derivative there~~

A function is analytic in an open region R if it is differentiable at all points $z \in R$. Also called "regular".
(A)

Quantization of Energy levels

Consider two-state Hamiltonian

$$\mathcal{H}_0 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

Time independent Schrödinger equation

$$\mathcal{H}_0 |\psi\rangle = E |\psi\rangle$$

Eigenstates : $|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Eigenvalues : $E_1 = a$, $E_2 = b$

Interaction with external field : $\mathcal{H}_1 = z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

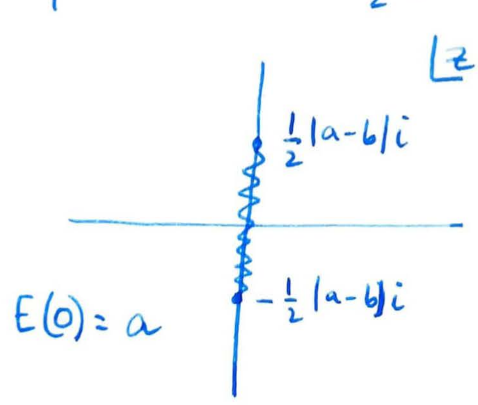
models electric charge etc and field strength

$$\mathcal{H} = \begin{pmatrix} a & z \\ z & b \end{pmatrix}$$

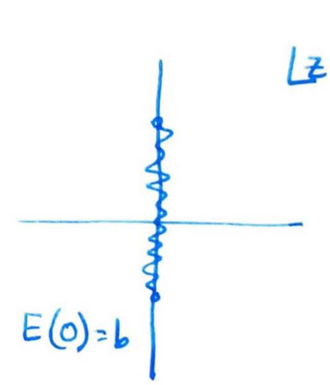
$$\mathcal{H}|\psi\rangle = E|\psi\rangle$$

$$\begin{vmatrix} a-E & z \\ z & b-E \end{vmatrix} = 0 \Rightarrow E(z) = \frac{a+b}{2} \pm \sqrt{\left(\frac{a-b}{2}\right)^2 + z^2}$$

$E(z)$ defined on 2-sheeted Riemann surface with branch points at $\pm \frac{1}{2}(a-b)i$



Sheet 1



Sheet 2

Path 1 $\Delta z = \Delta x$ (real)

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x,y) - iv(x,y)}{\Delta x}$$

$$= u_x(x,y) + i v_x(x,y)$$

Path 2 $\Delta z = i\Delta y$ (purely imaginary)

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - u(x,y) - iv(x,y)}{i\Delta y}$$

$$= -iu_y(x,y) + v_y(x,y)$$

Uniqueness $\Rightarrow$ $\boxed{u_x = v_y \ \& \ u_y = -v_x \quad \text{Cauchy-Riemann Conditions}}$

Taking another derivative on both sides of above equations, we get,

(A) $$\left.\begin{aligned} u_{xx} + u_{yy} &= 0 \\ \text{and } v_{xx} + v_{yy} &= 0 \end{aligned}\right\}$$ u & v are "harmonic" functions (satisfy Laplace's equation)

← Lec. Hrs 3.3

Points ~~of~~ where $f(z)$ is not differentiable are called <u>singular points</u> or singularities of $f(z)$

Real & Imaginary parts of an analytic function ~~are~~ form a conjugate pair of harmonic functions.

Ex: Given $u(x,y) = x^2 - y^2$, find $f(z)$ if it is analytic

Check harmonicity: $u_{xx} = 2$, $u_{yy} = -2$ ✓

$v_y = u_x = 2x \Rightarrow v(x,y) = 2xy + g(x)$

$v_x = -u_y = 2y = 2y + g'(x) \Rightarrow g'(x) = 0 \Rightarrow g(x) = c$

$\therefore f(z) = x^2 - y^2 + 2ixy + c = z^2 + c$

$$f(z) = R(z)\, e^{i\Theta(z)} \qquad z = x+iy = re^{i\theta}$$

$$\text{I)}\quad f'(z) = \lim_{\Delta x \to 0} \frac{R(x+\Delta x, y)\, e^{i\Theta(x+\Delta x, y)} - R(x,y)\, e^{i\Theta(x,y)}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[R(x,y) + \Delta x\, R_x(x,y) + O(\Delta x^2)\right] e^{i\left[\Theta(x,y) + \Delta x\, \Theta_x(x,y)\right]} - Re^{i\Theta}}{\Delta x}$$

[Notation: No arguments ⇒ arguments (x, y)]

$$= \lim_{\Delta x \to 0} \frac{(R + \Delta x R_x)\, e^{i\Theta}(1 + \Delta x\, i\Theta_x) - Re^{i\Theta}}{\Delta x}$$

$$= R_x e^{i\Theta} + iR\Theta_x e^{i\Theta}$$

$$\text{II)}\quad f'(z) = \lim_{\Delta y \to 0} \frac{R(x, y+\Delta y)\, e^{i\Theta(x, y+\Delta y)} - Re^{i\Theta}}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{(R + \Delta y R_y)\, e^{i\Theta}(1 + i\Delta y\, \Theta_y) - Re^{i\Theta}}{i\Delta y}$$

$$= R\Theta_y e^{i\Theta} - iR_y e^{i\Theta}$$

From I & II, $\boxed{R_x = R\Theta_y \quad \text{and} \quad R_y = -R\Theta_x}$

$$\text{III)}\quad f'(z) = \lim_{\Delta r \to 0} \frac{R(r+\Delta r, \theta)\, e^{i\Theta(r+\Delta r, \theta)} - Re^{i\Theta}}{\Delta r\, e^{i\theta}}$$

$$= \lim_{\Delta r \to 0} \frac{(R + \Delta r R_r)\, e^{i\Theta}(1 + i\Delta r\, \Theta_r) - Re^{i\Theta}}{\Delta r\, e^{i\theta}}$$

$$= R_r e^{i(\Theta - \theta)} + iR\Theta_r e^{i(\Theta-\theta)}$$

$$\text{IV)}\quad f'(z) = \lim_{\Delta \theta \to 0} \frac{R(r, \theta+\Delta\theta)\, e^{i\Theta(r, \theta+\Delta\theta)} - Re^{i\Theta}}{r i e^{i\theta}\, \Delta\theta}$$

$$= \lim_{\Delta \theta \to 0} \frac{(R + \Delta\theta R_\theta)\, e^{i\Theta}(1 + i\Delta\theta\, \Theta_\theta) - Re^{i\Theta}}{r i e^{i\theta}\, \Delta\theta}$$

$$= \frac{R\Theta_\theta}{r} e^{i(\Theta-\theta)} - i\frac{R_\theta}{r} e^{i(\Theta-\theta)}$$

From (III) & (IV),

$$\boxed{R_r = \frac{R\,\Theta_\theta}{r} \quad \text{and} \quad R_\theta = -r R\,\Theta_r}$$

Log $\quad f(z) = \log z \qquad z = re^{i\theta}$

$$= \log r + i\theta$$

Riemann surface → Infinite number of sheets each of which map to one horizontal strip of width 2π in the w-plane

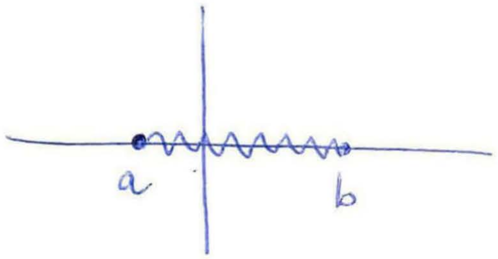
For the k^{th} sheet, $\quad 2\pi k < \theta \leq 2\pi(k+1)$

$$k \in \mathbb{Z}$$

$\sqrt{(z-a)(z-b)}$ $\qquad$ For $|z| \to \infty$, $f(z) \sim z$ or $f(z) \sim -z$

No branch point at ∞.

2-sheeted Riemann surface



Example of disconnected sheets

$$f(z) = i^z = e^{z \log i} = e^{z \log e^{i\pi/2 + 2n\pi i}}$$

$$= e^{iz(2n+1/2)\pi}$$

There is no branch point, no branch cut

Ex: (From NET book, pg 132, 10)

1. Show that $u(x,y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic. Determine the analytic function $f(z)$ whose real part $u(x,y)$ is.

Analytic / Regular / Holomorphic $\Rightarrow$ Differentiable and single-valued (at least on a Riemann sheet) at all points in a region R.

Complex Path Integrals / Contour Integrals

$$\int_C dz\, f(z)$$

Example 1: Find $\int_C |z|\, dz$ where c is the path that runs from $z=-1$ to $z=+1$ along (a) Real axis (b) Unit circle clockwise.

a) $\int_C |z|\, dz = \int_{-1}^{1} |t|\, dt$ $\quad z=t,\ t\in\mathbb{R}$

$= 2\int_0^1 t\,dt = 1$

b) $z=e^{i\theta}$ $\quad \theta : \pi \to 0$ $\quad dz = ie^{i\theta} d\theta$

$\Rightarrow \int_C |z|\, dz = \int_{\pi}^{0} ie^{i\theta} d\theta = e^{i\theta}\Big|_{\pi}^{0} = 1+1 = 2$

Example 2: Find $\int_C z^2 dz$ along same paths as above. (Ans. $= 2/3$)

Cauchy's Theorem: If $f(z)$ is analytic in a simply connected region that bounds a closed curve C, then, $\oint_C f(z)\, dz = 0$

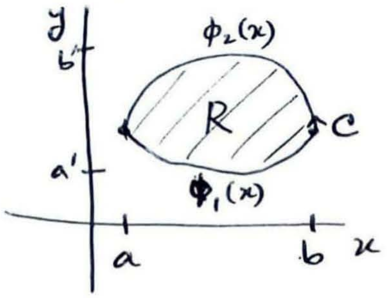
Consequence: $\int_{z_1}^{z_2} f(z)\, dz$ is path independent for all paths that connect z_1 and z_2 and $f(z)$ is analytic in a simply connected region that bounds these paths.

Entire function: Analytic $\forall\ z$ with $|z|<\infty$

If $f(z)$ is analytic $\forall z$ (including $z=\infty$), $f(z) = \text{constant}$.

Simply connected region — ① Includes boundaries ② Any two points can be connected by a continuous path ③ Any closed curve can be smoothly shrunk to a point.

Proof of Cauchy's Theorem



$$I = \oint_C f(z)\,dz$$

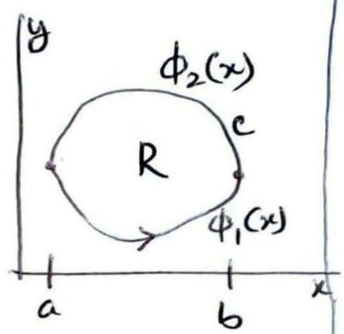
$$= \oint_C (dx + i\,dy)(u + iv)$$

Green's Theorem:

$$\oint_C \left[dx\, P(x,y) + dy\, Q(x,y)\right] = \int_R dx\,dy\,(Q_x - P_y) \quad \cdots \text{①}$$

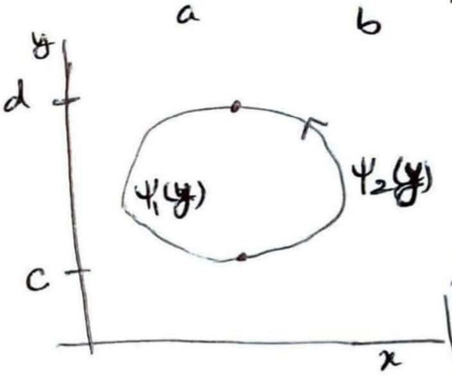
$$\int_a^b dx \int_{\phi_1(x)}^{\phi_2(x)} dy\, P_y = \int_a^b dx \left[P(x, \phi_2(x)) - P(x, \phi_1(x))\right]$$

$$= -\oint dx\, P(x,y)$$



$$\int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} dx\, Q_y = \int_c^d dy \left[Q(\psi_2(y), y) - Q(\psi_1(y), y)\right]$$

$$= \oint dy\, Q(x,y)$$



$\Rightarrow$ ①

$$\Rightarrow I = \oint \cancel{u\,dx}\left[(u + iv)dx + (iu - v)dy\right]$$

$$= \int_R dx\,dy\left[iu_x - v_x - u_y - iv_y\right]$$

$= 0$ by Cauchy Riemann Conditions.

Morera's Theorem : Path independence of integrals in a simply connected region R $\Rightarrow$ ~~and~~ analyticity in R.

Cauchy's Integral Theorem

$$f(z_0) = \frac{1}{2\pi i} \oint_C dz \, \frac{f(z)}{z - z_0}$$

$f(z)$ analytic in simply connected region around C

$$\oint_C \frac{dz}{z} = \log z \Big|_{z=1 \text{ on sheet } n}^{z=1 \text{ on sheet } n+1} = 2\pi(n+1)i - 2\pi n i = 2\pi i$$

C = unit ○ around origin

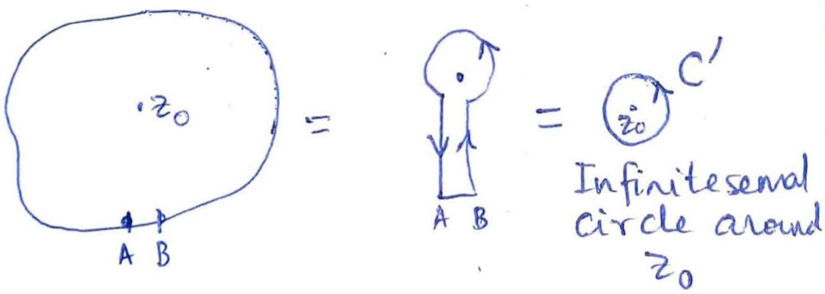
$$\Rightarrow \oint \frac{dz}{z - z_0} = 2\pi i$$

Proof:

$$I = \frac{1}{2\pi i} \underbrace{\oint_C dz \, \frac{f(z_0)}{z - z_0}}_{I_1} + \underbrace{\frac{1}{2\pi i} \oint_C dz \, \frac{f(z) - f(z_0)}{z - z_0}}_{I_2}$$

$$I_1 = f(z_0) \times \frac{1}{2\pi i} \oint_C \frac{dz}{z - z_0} = f(z_0)$$

For I_2:



On C', $\dfrac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$

$$\Rightarrow I_2 = \frac{f'(z_0)}{2\pi i} \oint_{C'} dz$$

$$= 0 \quad \because \text{Length of } C' \to 0$$

$$\Rightarrow I_2 = 0 \quad \& \quad I = f(z_0) \; ✓$$

$$\left| \int_C dz \, f(z) \right| \leq \int_C |dz| \, |f(z)| \leq (\text{Length of path } C) \times \max_C |f(z)|$$

Problems from NET Book

① Find the residue of the following functions at ~~x~~ infinity

a) $f(z) = \dfrac{z}{(z-a)(z-b)}$

b) $f(z) = \dfrac{z^3 - z^2 + 1}{z^3}$

② Given $\int_0^\infty e^{-(a+ib)x} dx = \dfrac{1}{a+ib}$, $a>0$, Show $\int_0^\infty \dfrac{\sin bx}{x} dx = \dfrac{\pi}{2}$

③ (a) Find rad. of convergence of the series $\sum_{n=1}^\infty \dfrac{n!}{n^n} x^n$

(b) Prove by method of residues $\int_{-\infty}^\infty \dfrac{\cos mx \, dx}{(x+c)^2 + a^2} = \dfrac{\pi}{a} e^{-a|m|} \cos cm$
$a, c \in \mathbb{R}$ and $a>0$

~~$f(z_0) = \oint$~~

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{z - z_0}$$

$f(z)$ analytic in simply connected region around C.

$$\Rightarrow f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^2}, \quad f''(z_0) = \frac{2}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^3},$$

$$\cdots \quad \boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}}$$

$\Rightarrow$ Derivatives of all orders exist at z_0.

$\Rightarrow$ Existence of Taylor series expansion at z_0.

$$f(z) = \sum_{n=0}^\infty a_n (z-z_0)^n \quad \text{where} \quad a_n = f^{(n)}(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

Consider $f(z)$ which is analytic in the region $|z| < R$.

$$\Rightarrow f(z) = \sum_{n=0}^\infty a_n z^n \quad \text{where} \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{z^{n+1}}$$

$$\therefore |a_n| < \frac{1}{2\pi} \frac{2\pi r}{r^{n+1}} \max_{|z|=r} |f(z)|$$

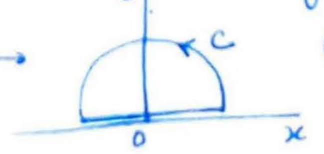
Choose C as a circular path of radius $r < R$ around 0.

$$\Rightarrow |a_n| < r^{-n} \max_{|z|=r} |f(z)| \qquad \text{(Cauchy's Inequality)}$$

For a bounded entire function, r can be chosen to be arbitrarily large $\Rightarrow a_n = 0$ for $n \geq 1$ for a bounded entire function $\Rightarrow$ Liouville's theorem: A bounded entire function must be a constant.

Using Cauchy's theorem to calculate an integral:

$$I = \oint_C \frac{dz}{z^2+1} \quad \text{where } C \text{ is} \rightarrow$$



(Flat part on the real axis)

$$= \oint_C \frac{dz}{(z+i)(z-i)}$$

$$= \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-i}\, dz \quad \text{where } f(z) = \frac{2\pi i}{z+i}$$

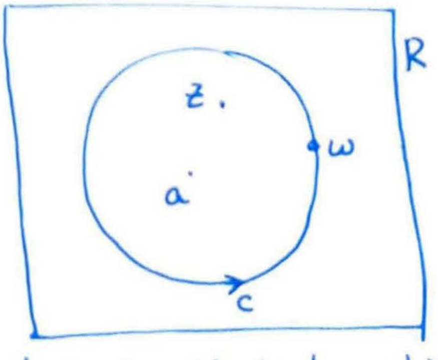
which is analytic in a ~~reg~~ simply connected region containing C

$$= f(i)$$

$$= \pi$$

Derivation of Taylor Series:

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\omega)}{\omega - z}\, d\omega$$



$f(z)$ is analytic in R

Choose C to be a circular contour centered at 'a' and of radius $r = |\omega - a|$ such that $|\omega - a| > |z - a|$

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\omega)\, d\omega}{(\omega - a)(1 - \varepsilon)} \quad \text{where } \varepsilon = 1 - \frac{\omega - z}{\omega - a} = \frac{z - a}{\omega - a}$$

$$\frac{1}{1-\varepsilon} = 1 + \varepsilon + \varepsilon^2 + \dots + \varepsilon^N + \frac{\varepsilon^{N+1}}{1-\varepsilon}$$

Term-by-term operations

Within its circle of ~~o~~ convergence it is okay to differentiate and integrate a Taylor series term-by-term. If we have another Taylor series which converges in that circle, it is ~~oky a~~ okay to add the two series or multiply them together.

$$f_1(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \;,\quad f_2(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

$$f_1(z) + f_2(z) = \sum_{n=0}^{\infty} (a_n + b_n)(z-z_0)^n$$

$$f_1(z) f_2(z) = \sum_{n_1=0}^{\infty} a_{n_1} (z-z_0)^{n_1} \sum_{n_2=0}^{\infty} b_{n_2} (z-z_0)^{n_2}$$

$$= \sum_{n=0}^{\infty} c_n (z-z_0)^n \;,\quad c_n = \sum_{m=0}^{n} a_m b_{n-m}$$

$$f_1'(z) = \sum_{n=1}^{\infty} n\, a_n (z-z_0)^{n-1}$$

$$\int f_1(z)\, dz = \text{constant} + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-z_0)^{n+1}$$

If $a_0 \neq 0$,

$$\frac{1}{f_1(z)} = \frac{1}{a_0 + \sum_{n=1}^{\infty} a_n (z-z_0)^n} = \frac{1}{a_0} \left[1 + \frac{1}{a_0} \sum_{n=1}^{\infty} a_n (z-z_0)^n \right]^{-1}$$

$$= \frac{1}{a_0} - \frac{a_1}{a_0^2}(z-z_0) + \frac{a_1^2 - a_0 a_2}{a_0^3}(z-z_0)^2 + \frac{2a_0 a_1 a_2 - a_1^3 - a_0^2 a_3}{a_0^4} \times (z-z_0)^3 + \dots$$

Inverting a function, in general, results in a change in the radius of convergence (introduces new singularities).

Uniqueness of Taylor series

Suppose $f(z)$ is represented by two Taylor series around the point $z = z_0$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{and} \quad f(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

Differentiating the first series m times and setting $z = z_0$, we get,

$$\cancel{f(z_0)} \; f^{(m)}(z_0) = m! \, a_m$$

Similarly, from the second series, we get,

$$f^{(m)}(z_0) = m! \, b_m$$

Thus, $a_m = b_m \ \forall m \Rightarrow$ Taylor series is unique.

$\Rightarrow$ If we can find a convergent series that is of the form $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ representing the function around $z = z_0$, then it must be its Taylor series about that point.

↳ The same is true for a Laurent series in its region of convergence.

Regions of convergence of a Taylor series from tests of convergence of series

$$S = \sum_{n=0}^{\infty} A_n \quad , \qquad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

① Ratio test: Series S converges absolutely if

$$\lim_{n\to\infty} \left| \frac{A_{n+1}}{A_n} \right| < 1$$

and diverges if $\displaystyle \lim_{n\to\infty} \left| \frac{A_{n+1}}{A_n} \right| > 1$

⇒ Taylor series converges if

$$\lim_{n\to\infty} \left| \frac{a_{n+1} (z-z_0)^{n+1}}{a_n (z-z_0)^n} \right| < 1$$

$$\Rightarrow |z - z_0| < r \quad , \quad \text{where } r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

---

② Root Test:

$$\lim_{n\to\infty} |A_n|^{1/n} \begin{cases} <1 \Rightarrow \text{convergence} \\ >1 \Rightarrow \text{divergence} \end{cases}$$

~~Thus~~ Taylor series converges if

$$\lim_{n\to\infty} \left| a_n (z-z_0)^n \right|^{1/n} < 1$$

$$\Rightarrow \quad |z - z_0| < r = \lim_{n\to\infty} |a_n|^{-1/n}$$

$$\Rightarrow f(z) = \frac{1}{2\pi i}\oint \frac{f(\omega)}{\omega - a}\left[1 + \varepsilon + \varepsilon^2 + \cdots + \varepsilon^N + \frac{\varepsilon^{N+1}}{1-\varepsilon}\right] d\omega$$

$$= \sum_{n=0}^{N} (z-a)^n \frac{1}{2\pi i}\oint \frac{f(\omega)\, d\omega}{(\omega - a)^{n+1}} + R_N$$

$$= \sum_{n=0}^{N} (z-a)^n \frac{f^{(n)}(a)}{n!} + R_N$$

$$R_N = \frac{1}{2\pi i}\oint \frac{f(\omega)\, \varepsilon^{N+1}}{\omega - z} d\omega$$

$$\Rightarrow |R_N| \leqslant \frac{1}{2\pi}\left(\max_C \frac{|f(\omega)|}{|\omega - z|}\right) 2\pi r\, |\varepsilon|^{N+1}$$

$$\leqslant \cancel{\frac{1}{2\pi}} \frac{\max_C |f(\omega)|}{\min_C |\omega - z|} r\, |\varepsilon|^{N+1}$$

$$= \frac{\max_C |f(\omega)|}{r - |z-a|} r\, |\varepsilon|^{N+1} \longrightarrow 0 \text{ as } N \to \infty$$

$$\Rightarrow R_N \to 0 \text{ as } N \to \infty \Rightarrow f(z) = \sum_{n=0}^{\infty} (z-a)^n \frac{f^{(n)}(a)}{n!}$$

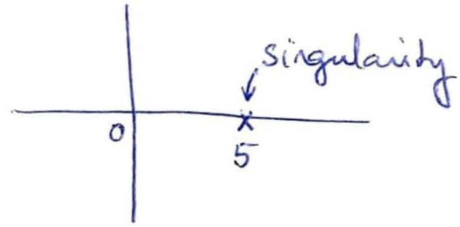
The above derivation works as long as the point z is closer to the point a, than the nearest singularity of $f(z)$ to point 'a'. Thus, the radius of convergence of a Taylor series is the distance from the nearest singularity.

[Analytic continuation (not in syllabus): Going around a singularity using ~~to Taylor series~~ a sequence of Taylor series centered at ~~vart~~ varying points.

Laurent Series: $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \to$ Needs $f(z)$ to be analytic in an annular region between 2 concentric circles centered at z_0.

Laurent Series Examples

① Expand $f(z) = \dfrac{z}{(z-5)^5}$ about $z=5$ in a Laurent series.



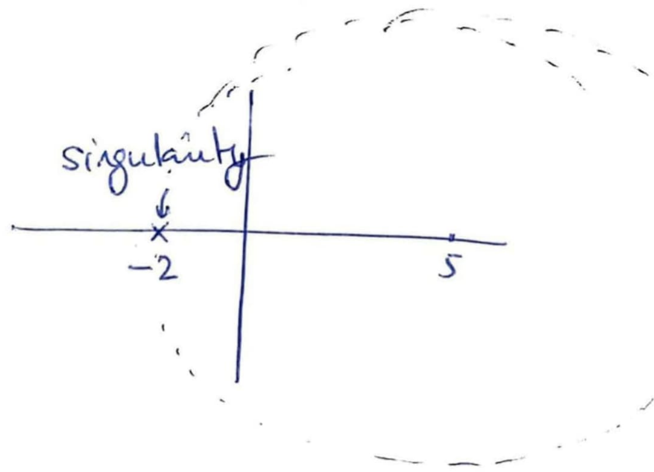
$$f(z) = \frac{z}{(z-5)^5}$$

$$= \frac{z-5+5}{(z-5)^5}$$

$$= \frac{5}{(z-5)^5} + \frac{1}{(z-5)^4}$$

Region of convergence:
$0 < |z-5| < \infty$

② $\dfrac{z}{z+2}$ about $z=5$



$$\frac{z}{z+2} = \frac{z-5+5}{z-5+7}$$

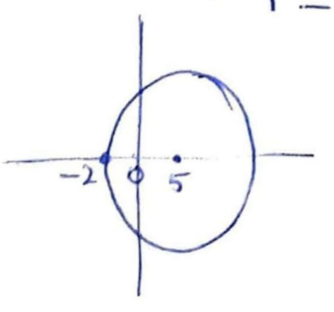
$$= \frac{z-5+5}{(z-5)\left[1+\frac{7}{z-5}\right]}$$

$$= \frac{(z-5)+5}{z-5}\left[1 - \frac{7}{z-5} + \frac{49}{(z-5)^2} - \frac{343}{(z-5)^3} + \dots\right]$$

$$= \left(1 + \frac{5}{z-5}\right)\left(1 - \frac{7}{z-5} + \frac{49}{(z-5)^2} - \frac{343}{(z-5)^3} + \dots\right)$$

$$= 1 + \frac{5-7}{z-5} + \frac{49-35}{(z-5)^2} + \frac{49\times5-343}{(z-5)^3} + \dots$$

$$= 1 - \frac{2}{z-5} + \frac{14}{(z-5)^2} - \cancel{98}\,\frac{98}{(z-5)^3} + \dots$$



Region of convergence: $7 < |z-5| < \infty$

Note: The same function is analytic in the circle $|z-5|<7$ and must have a Taylor series in that region.

$$\frac{z}{z+2} = \frac{z-5+5}{7+z-5}$$

$$= \frac{z-5+5}{7}\left[1+\frac{z-5}{7}\right]^{-1}$$

$$= \left(\frac{5}{7}+\frac{z-5}{7}\right)\left(1-\frac{z-5}{7}+\frac{(z-5)^2}{49}-\frac{(z-5)^3}{343}\cdots\right)$$

$$= \frac{5}{7}+\left(\frac{1}{7}-\frac{5}{49}\right)(z-5)+\left(-\frac{1}{49}+\frac{5}{343}\right)(z-5)^2+\cdots$$

$$= \frac{5}{7}+\frac{2}{49}(z-5)-\frac{2}{343}(z-5)^2+\cdots$$

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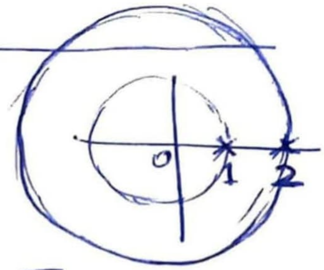
③ $f(z) = \dfrac{1}{(z-1)(2-z)}$ around $z=0$



(a) $|z|<1 \Rightarrow f(z)$ is analytic $\Rightarrow$ Taylor series

$$f(z) = -\frac{1}{2}(1-z)^{-1}\left(1-\frac{z}{2}\right)^{-1}$$

$$= -\frac{1}{2}(1+z+z^2+\cdots)\left(1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\cdots\right)$$

$$= -\frac{1}{2}-\frac{1}{2}\left(1+\frac{1}{2}\right)z-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{4}\right)z^2-\cdots$$

$$= -\frac{1}{2}-\frac{3}{4}z-\frac{7}{8}z^2 \cdots\cdots$$

(b) $1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)$$

$$= \frac{1}{2z}\Big[\cdots + \frac{1}{z^2}\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)$$
$$+ \frac{1}{z}\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right)$$
$$+ \left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)$$
$$+ z\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)$$
$$+ z^2\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)+\cdots\Big]$$

$$= \frac{1}{2z}\left[\cdots + \frac{2}{z^2}+\frac{2}{z}+2+z+\frac{z^2}{2}+\cdots\right]$$

$$= \cdots + \frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}+\frac{1}{2}+\frac{z}{4}+\cdots$$

(c) $|z| > 2 \Rightarrow \left|\frac{1}{z}\right| < 1$ & $\left|\frac{2}{z}\right| < 1$

$$f(z) = \frac{-1}{z^2}\left(1-\frac{1}{z}\right)^{-1}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{-1}{z^2}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\right)$$

$$= -\left[\frac{1}{z^2}+\frac{3}{z^3}+\frac{7}{z^4}+\cdots\right] = -\frac{1}{z^2}-\frac{3}{z^3}-\frac{7}{z^4}-\cdots$$

Method 2 (Using Partial Fractions)

$$f(z) = \frac{1}{(z-1)(2-z)} = \frac{1}{z-1} + \frac{1}{2-z}$$

For $|z|<1$, $\frac{1}{z-1} = -1(1-z)^{-1} = -1 - z - z^2 - z^3 - z^4 - \cdots$ (i)

For $|z|>1$, $\frac{1}{z-1} = \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$ (ii)

For $|z|<2$, $\frac{1}{2-z} = \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} = \frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \frac{z^3}{16} + \cdots$ (iii)

For $|z|>2$, $\frac{1}{2-z} = -\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} = -\frac{1}{z} - \frac{2}{z^2} - \frac{4}{z^3} - \frac{8}{z^4} - \cdots$ (iv)

$\therefore$ For $|z|<1$, using (i) and (iii), we get,

$$f(z) = -\frac{1}{2} - \frac{3z}{4} - \frac{7z^2}{8} - \frac{15z^3}{16} - \cdots$$

For $1<|z|<2$, using (ii) and (iii), we get,

$$f(z) = \cdots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \frac{z^3}{16} + \cdots$$

For $|z|>2$, using (ii) & (iv), we get,

$$f(z) = -\frac{1}{z^2} - \frac{3}{z^3} - \frac{7}{z^4} - \cdots$$

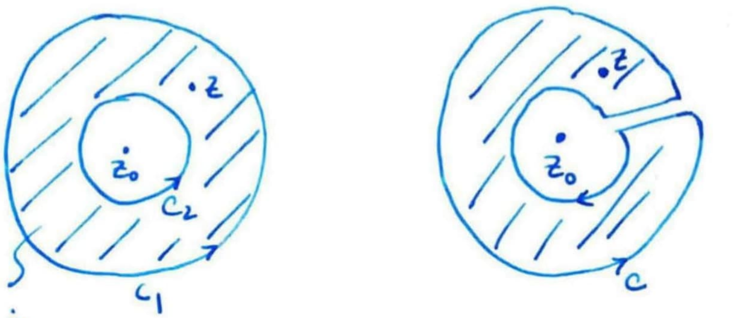
MORAL: It is easier to add two similar series than to multiply them. Breaking a complicated ~~fun~~ function into a sum of simpler functions may be "better" than breaking it into a product of simpler functions

$$\frac{f(z)}{(z-z_0)^{n+1}} = \sum_{m=-\infty}^{\infty} a_m (z-z_0)^{m-n-1}$$

$$= \sum_{p=-\infty}^{\infty} a_{n+p+1} (z-z_0)^{p}$$

$$\Rightarrow \quad a_n = \frac{1}{2\pi i} \oint_{C'} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \,,$$ C' is any closed curve encircling z_0 and lying in the region where $f(z)$ is analytic.

Derivation :



$f(z)$ is analytic here.

$$\frac{1}{2\pi i} \oint_{C} \frac{f(\omega)\, d\omega}{(\omega - z)} = f(z)$$

$$\Rightarrow f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\omega)\, d\omega}{\omega - z} + \frac{1}{2\pi i} \oint_{C_2} \frac{f(\omega)\, d\omega}{z - \omega}$$

$$= \frac{1}{2\pi i} \oint_{C_1} \frac{f(\omega)\, d\omega}{(\omega - z_0)(1-\varepsilon_1)} + \frac{1}{2\pi i} \oint_{C_2} \frac{f(\omega)\, d\omega}{(z-z_0)(1-\varepsilon_2)}$$

where $\varepsilon_1 = \dfrac{z-z_0}{\omega - z_0}$

$\varepsilon_2 = \dfrac{\omega - z_0}{z - z_0}$

$$= \frac{1}{2\pi i} \oint_{C_1} \frac{f(\omega)}{(\omega - z_0)} \sum_{n=0}^{N_1} \varepsilon_1^{\,n}\, d\omega + \frac{1}{2\pi i} \oint_{C_2} \frac{f(\omega)}{(z-z_0)} \sum_{n=0}^{N_2-1} \varepsilon_2^{\,n}\, d\omega + R_1^{(N_1)} + R_2^{(N_2)}$$

$$= \sum_{n=-N_2}^{N_1} \frac{(z-z_0)^n}{2\pi i} \oint_{D} \frac{f(\omega)\, d\omega}{(\omega - z_0)^{n+1}} + R_1^{(N_1)} + R_2^{(N_2)}$$

where D is any contour around z_0 in the region where $f(z)$ is analytic

$$R_1^{(N_1)} = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\omega)}{\omega - z_0} \frac{\varepsilon_1^{N_1+1}}{1-\varepsilon_1} \, d\omega$$

$$= \frac{1}{2\pi i} \oint \frac{f(\omega)}{\omega - z} \varepsilon_1^{N+1} \, d\omega$$

$$\Rightarrow \left| R_1^{(N_1)} \right| \leqslant \frac{1}{2\pi} \oint \frac{|f(\omega)|}{|\omega - z|} |\varepsilon_1|^{N_1+1} |d\omega| \to 0 \Rightarrow R_1^{(N_1)} \to 0 \text{ as } N_1 \to \infty$$

$$R_2^{(N_2)} = \frac{1}{2\pi i} \oint_{C_2} \frac{f(\omega)}{z - z_0} \frac{\varepsilon_2^{N_2}}{1-\varepsilon_2} \, d\omega$$

$$\Rightarrow \left| R_2^{(N_2)} \right| = \frac{1}{2\pi} \oint \frac{|f(\omega)|}{|(z - \omega)|} |\varepsilon_2|^{N_2} |d\omega| \to 0 \quad \text{as } N_2 \to \infty$$

$$\Rightarrow R_2^{(N_2)} \to 0 \quad \text{as } N_2 \to \infty$$

$\Rightarrow$ Laurent series converges in an annulus around z_0 whose inner and outer radii are optimized so that the function is analytic in the annular region and has singularities just inside the inner circle & ~~jo~~ just outside the outer circle.

If $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ in some region around z_0,

$$\oint_C f(z) dz = 2\pi i \, a_{-1}$$

a_{-1} is called the residue of $f(z)$ at z_0.

Classification of singularities based on Laurent expansion

If inner radius of annular region where the ~~two~~ Laurent series converges has zero radius, the singularity is said to be an isolated singularity.

~~If $f(z) \neq \sum_{n=-N}^{\infty} f(z) d$~~

If $f(z) = \sum_{n=-N}^{\infty} a_n (z-z_0)^n$, $a_{-N} \neq 0$

a) $N \cancel{\leq} 0 \rightarrow$ Removable singularity. (Needs redefinition of $f(z_0)$)

b) $N = 1 \rightarrow$ Simple pole, Residue, $a_{-1} = \lim_{z \to z_0} f(z)(z-z_0)$

c) $N \geq 2 \rightarrow$ Pole of order N.

d) $N = \infty \rightarrow$ Essential singularity

Classification of point at infinity $\Leftrightarrow$ Classification of the point $z=0$ for the function $g(z) = f(1/z)$

Unique ~~ness~~ property of $z=\infty$: Can have nonzero residue even when analytic:

$$\oint_{\substack{C \\ \text{around} \\ \infty}} f(z)\, dz = \oint_{\substack{C \\ \text{around} \\ \text{zero}}} f(1/z')(-1/z'^2)\, dz'$$

$$\Rightarrow \boxed{\text{Residue of } f(z) \text{ at } \infty = \text{Residue of } \frac{-f(1/z)}{z^2} \text{ at } z=0}$$

<u>Residue at poles</u>

$$f(z) = \sum_{n=-N}^{\infty} a_n (z-z_0)^n$$

$$\Rightarrow f(z)(z-z_0)^N = \sum_{n=-N}^{\infty} a_n (z-z_0)^{n+N} = \sum_{p=0}^{\infty} a_{p-N} (z-z_0)^p$$

a_{-1} is the coefficient of ~~N-1~~ $(z-z_0)^{N-1}$ in above Taylor series

$$\Rightarrow \boxed{a_{-1} = \frac{1}{(N-1)!} \lim_{z \to z_0} \frac{d^{N-1}}{dz^{N-1}} \left[f(z)(z-z_0)^N \right]}$$

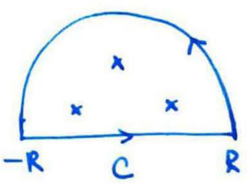
Residue theorem

If $f(z)$ is analytic in an area bounded by a contour C, except for a finite number of poles & isolated essential singularities, then

$$\oint f(z)\,dz = 2\pi i \sum \text{Residues}$$

Example: $\displaystyle\int_0^\infty \frac{x^2\,dx}{x^6+1} = ?$

$$I = \int_0^\infty \frac{x^2\,dx}{x^6+1} = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2\,dx}{x^6+1}$$



$$\oint \frac{z^2\,dz}{z^6+1} = 2I + \int_{\frown} \frac{z^2\,dz}{z^6+1}$$

$$I_2 = \int_{\frown} \frac{z^2\,dz}{z^6+1}\quad ,\quad I_0 = \oint \frac{z^2\,dz}{z^6+1}$$

$$|I_2| \le \int_{\frown} \frac{|z|^2|dz|}{|z^6+1|} = R^2 \int_{\frown} \frac{|dz|}{|z^6+1|} \le \frac{R^2\,\pi R}{\min|z^6+1|}$$

$$\cancel{|z^6+1| > ||z^6|-1| = R-1}$$

$$|z^6+1| > \left||z|^6-1\right| = R^6-1$$

$$\Rightarrow\ |I_2| \le \frac{\pi R^3}{R^6-1} \to 0 \text{ as } R\to\infty$$

$$\Rightarrow\ I = \tfrac{1}{2} I_0$$

z^6+1 has simple poles at $z_k = e^{i\pi/6 + 2k\pi i/6} = e^{i\frac{\pi}{6}(2k+1)}$

$k = 0, 1, 2, \ldots, 5$

Poles inside C: $e^{i\pi/6}, e^{i\pi/2}, e^{5i\pi/6}$

Residue at $z_k = \displaystyle\lim_{z\to z_k} (z-z_k)\frac{z^2}{z^6+1} = \lim_{z\to z_k} \frac{3z^2-2zz_k}{6z^5} = \frac{1}{6z_k^3}$

$$\Rightarrow I = \frac{1}{2}\times\frac{2\pi i}{6}\left[\frac{1}{e^{i\pi/2}} + \frac{1}{e^{3i\pi/2}} + \frac{1}{e^{5i\pi/2}}\right] = \frac{\pi}{6}$$

② Find residues at all poles in finite complex plane for

(a) $\dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ (b) $e^z \operatorname{cosec}^2 z$

(a) Poles at $\begin{cases} z=-1 & \text{(2nd order)} \\ z=\pm 2i & \text{(Simple)} \end{cases}$

For the simple poles,

$$\text{Residue} = \lim_{z\to z_0} (z-z_0)\frac{z^2-2z}{(z+1)^2(z^2+4)} = \frac{(\pm 2i)^2-2(\pm 2i)}{(\pm 2i+1)^2(\cancel{z \pm 2i} \pm 2i \pm 2i)}$$

$$= \frac{-4 \mp 4i}{(-4+1 \pm 4i)(\pm 4i)} = \frac{(-1 \pm i)}{(-3 \pm 4i)}$$

$$= \frac{(-1 \pm i)(-3 \mp 4i)}{25}$$

$$= \frac{+3+4 \mp 3i \pm 4i}{25} = \frac{7 \pm i}{25}$$

For $z=-1$

$$\text{Residue} = \lim_{z\to -1} \frac{d}{dz}\left[(z+1)^2\left(\frac{z^2-2z}{(z^2+4)(z+1)^2}\right)\right]$$

$$= \lim_{z\to -1} \frac{d}{dz} \frac{z^2-2z}{z^2+4}$$

$$= \lim_{z\to -1} \frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2}$$

$$= \frac{-20-(3)(-2)}{25} = -\frac{14}{25}$$

(b) $f(z) = \dfrac{e^z}{\sin^2 z}$

~~Simple poles at zeros of sin z~~

Poles of order 2 at zeros of $\sin z$: $z = m\pi \quad m \in \mathbb{Z}$

let $z - m\pi = u$

$\Rightarrow \sin z = \sin(u+m\pi) = (-1)^m \sin u$

$\Rightarrow \sin^2 z = \sin^2 u$

$$\Rightarrow f(z) = \frac{e^z}{\sin^2 z} = e^{m\pi} e^{u} (\sin u)^{-2}$$

$$= e^{m\pi}\left(1+u+\frac{u^2}{2!}+\cdots\right)\left(u-\frac{u^3}{3!}+\frac{u^5}{5!}+\cdots\right)^{-2}$$

$$= \frac{e^{m\pi}}{u^2}\left(1+u+\frac{u^2}{2!}+\cdots\right)\left(1+\frac{u^2}{3}+\cdots\right)$$

$$= e^{m\pi}\left(\frac{1}{u^2}+\frac{1}{u}+\cdots\right)$$

$\Rightarrow$ Residue at $z = m\pi$ is $e^{m\pi}$

3 $\displaystyle\oint_C \frac{dz}{z\sin z}$ C: Circle encircling origin

$$\frac{1}{z\sin z} = \frac{1}{z^2\left(1-\frac{z^2}{2!}+\frac{z^4}{5!}-\cdots\right)}$$

$$= \frac{1}{z^2}\left[1+\left(z^2-\frac{z^4}{5!}+\cdots\right)+\left(z^2-\frac{z^4}{5!}+\cdots\right)^2+\cdots\right]$$

$$= \frac{1}{z^2}\left[1+z^2+\frac{119}{120}z^4+\cdots\right] = \sum_{n=-\infty}^{\infty} a_n z^n$$

$\Rightarrow$ Residue $= a_{-1} = 0$

$$\Rightarrow \oint \frac{dz}{z\sin z} = 0$$

[Second Method: $\frac{1}{z\sin z}$ is even $\Rightarrow$ For closed circular contour around origin, for every z, there is a contribution from $-z$ whose contribution cancels the contribution from z to the integral]

4 $\displaystyle\oint_C \frac{e^z\, dz}{z(z-1)}$ C: Contour of Radius 2 around origin

Simple poles at $z=0$ and $z=1$

Residue at $z=0$: $\lim_{z\to 0} \frac{e^z}{z-1} = -1$

Residue at $z=1$: $\lim_{z\to 1} \frac{e^z}{z} = e$

$$\Rightarrow \oint_c \frac{e^z\,dz}{z(z-1)} = 2\pi i(e-1)$$

Integrals of the form $I = \int_0^{2\pi} d\theta\, f(\sin\theta, \cos\theta)$

$$z = e^{i\theta} \Rightarrow dz = i e^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

$$\Rightarrow I = \frac{1}{i}\oint_C \frac{dz}{z} f\left(\frac{z-1/z}{2i}, \frac{z+1/z}{2}\right)$$

C: unit circle around origin

Examples:

① $I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \oint_C \frac{dz/iz}{2+\frac{z+1/z}{2}}$

$$= \frac{2}{i}\oint \frac{dz}{z^2+4z+1}$$

Simple poles at $\frac{-4\pm\sqrt{16-4}}{2} = -2\pm\sqrt{3}$

Only $z=-2+\sqrt{3}$ is inside the unit circle

$$\Rightarrow I = 2\pi i \times \frac{2}{i} \times \frac{1}{(-2+\sqrt{3})-(-2-\sqrt{3})} = \frac{2\pi}{\sqrt{3}}$$

① $\frac{1}{2\pi i}\oint_C \frac{e^{tz}}{z^2(z^2+2z+2)}$ $\quad C : |z|=3$

Ans. $\frac{t-1}{2} + \frac{e^{-t}\cos t}{2}$

② $\int_0^\infty \frac{dx}{x^6+1}$ Ans. $2\pi/3$

③ $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)}$ Ans. $\frac{7\pi}{50}$

④ $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$ Ans. $\pi/2$

⑤ $\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$ Ans. π

⑥ $\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2}$ Ans. $5\pi/32$

<u>Cauchy Principal Value</u> of the real integral of $f(x)$ with an isolated singularity at x_0 is

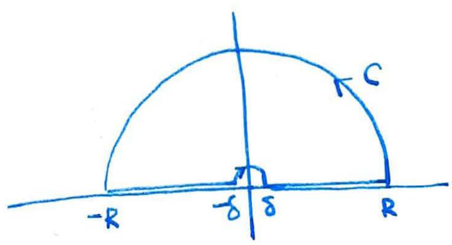
$$P\int_a^b f(x)\,dx \quad \text{or} \quad \fint f(x)\,dx \equiv \lim_{\delta\to 0^+}\left[\int_a^{x_0-\delta} f(x)\,dx + \int_{x_0+\delta}^{b} f(x)\,dx\right]$$

Example: $I = \int_0^\infty \frac{\sin x}{x}\,dx$

$$= \int_0^\infty \frac{e^{ix}-e^{-ix}}{2ix}\,dx$$

$$= \frac{1}{2i}\int_0^\infty \left[\frac{e^{ix}}{x} + \frac{e^{-ix}}{-x}\right]dx$$

$$= \frac{1}{2i}\,P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx$$



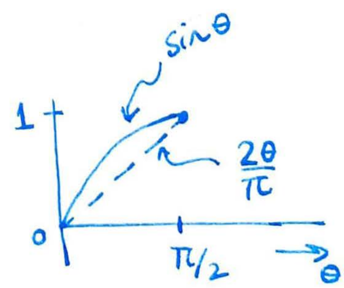
$$\oint \frac{e^{iz}\,dz}{z} = 0$$

$$= 2iI + I_\delta + I_R$$

$$I_\delta = \lim_{\delta\to 0}\int_\pi^0 \frac{e^{i\delta e^{i\theta}}}{\delta e^{i\theta}}\,\cancel{d(\delta e^{i\theta})}\,\delta i e^{i\theta}\,d\theta$$

$$= -i\lim_{\delta\to 0}\int_0^\pi e^{i\delta e^{i\theta}}\,d\theta$$

$$= \cancel{i/\pi}\; -i\int_0^\pi d\theta = -i\pi$$



$$I_R = \lim_{R\to\infty}\int_0^\pi \frac{e^{iRe^{i\theta}}}{Re^{i\theta}}\,Rie^{i\theta}\,d\theta$$

$$= i\lim_{R\to\infty}\int_0^\pi e^{iR\cos\theta - R\sin\theta}\,d\theta$$

$$|I_R| \leq \lim_{R\to\infty}\int_0^\pi \cancel{1}\,e^{-R\sin\theta}\,d\theta \leq \lim_{R\to\infty}\int_0^\pi e^{-R2\theta/\pi}\,d\theta$$

$$\Rightarrow |I_R| \leq \lim_{R\to\infty} \frac{-\pi}{2R}\,e^{-2R\theta/\pi}\Big|_0^\pi = 0 \quad \Rightarrow I_R = 0$$

$$2iI - i\pi = 0 \Rightarrow I = \pi/2 \Rightarrow \boxed{\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}}$$

Jordan's lemma:

If $f(Re^{i\theta}) \to 0$ as $R \to \infty$ and $0 \leq \theta \leq \pi$, then

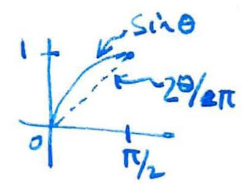
$$I_R \equiv \lim_{R\to\infty} \int_C e^{iaz} f(z)\,dz = 0$$

where a is a positive real number and C is a semicircular contour of radius R centered at origin

$$I_R = \lim_{R\to\infty} \int_0^\pi e^{iaR\cos\theta - aR\sin\theta} f(Re^{i\theta})\, Rie^{i\theta}\,d\theta$$

$$\Rightarrow |I_R| \leq \lim_{R\to\infty} \int_0^\pi e^{-aR\sin\theta} \left|f(Re^{i\theta})\right| R\,d\theta$$

$$\leq \lim_{R\to\infty} R\epsilon_R \int_0^\pi e^{-aR\sin\theta}\,\cancel{R}\,d\theta \qquad \text{where } \epsilon_R = \max_\theta \left|f(Re^{i\theta})\right|$$



$$\leq \lim_{R\to\infty} R\epsilon_R \int_0^\pi e^{-2aR\theta/\pi}\,d\theta$$

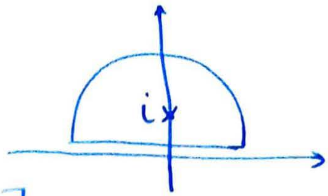
$$= \lim_{R\to\infty} R\epsilon_R \left. \frac{e^{-2aR\theta/\pi}}{-2aR/\pi} \right|_0^\pi$$

$$= \lim_{R\to\infty} \frac{\pi\epsilon_R}{2a}\left[1 - e^{-2aR}\right]$$

$$= 0 \qquad \because \lim_{R\to\infty} \epsilon_R = 0$$

$$\Rightarrow I_R = 0$$

① $$\int_{-\infty}^{\infty} dx \frac{\cos x}{x^2+1} = \mathrm{Re} \int_{-\infty}^{\infty} dx \frac{e^{ix}}{x^2+1}$$



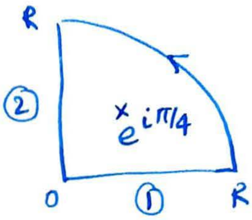
$$= \mathrm{Re} \left[\oint_C dz \frac{e^{iz}}{z^2+1} - I_R \right]$$

$I_R \to 0$ by Jordan's lemma

$$= \mathrm{Re} \left[2\pi i \frac{e^{i(i)}}{i+i} \right]$$

$$= \frac{\pi}{e}$$

② $$\int_0^{\infty} \frac{x\,dx}{x^4+1} = I_1 \cancel{+ I_2 + I_R}$$



$$I = \oint \frac{z\,dz}{z^4+1} = I_1 + I_2 + I_R$$

$$z^4+1 = (z^2+i)(z^2-i)$$
$$= (z^2+i)(z+e^{i\pi/4})(z-e^{i\pi/4})$$

$$I = 2\pi i \left(\text{Residue at } e^{i\pi/4} \right)$$

$$= 2\pi i \frac{e^{i\pi/4}}{2i \times 2e^{i\pi/4}} = \frac{\pi}{2}$$

$$I_R = \lim_{R\to\infty} \int R i e^{i\theta} d\theta \frac{Re^{i\theta}}{Re^{4i\theta}+1}$$

$$|I_R| \le \lim_{R\to\infty} R^2 \int \frac{d\theta}{|Re^{4i\theta}+1|}$$

$$\le \lim_{R\to\infty} R^2 \int \frac{d\theta}{R-1} \to 0 \quad \cancel{I_R \to 0 \text{ as } R\to\infty}$$

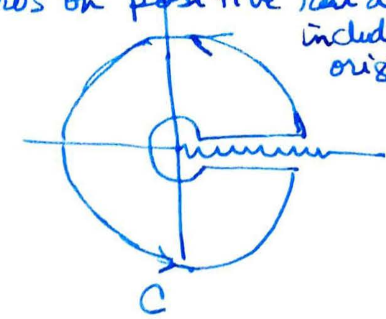
$\Rightarrow I_R = 0$

$$I_2 = \int_{\infty}^{0} \frac{iy\, i dy}{y^4+1} = \int_0^{\infty} \frac{y\,dy}{y^4+1} = I_1$$

$$\Rightarrow 2I_1 = \pi/2 \Rightarrow I_1 = \frac{\pi}{4}$$

Integrals of the form $\int_0^\infty dx \, \frac{P(x)}{Q(x)}$

Degree of Q at least 2 more than P & Q has no zeros on positive real axis including origin



Consider $J(\varepsilon, R) = \oint_C dz \, \frac{P(z) \log z}{Q(z)}$

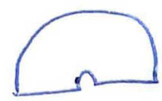
Contributions from circular parts $\to 0$

$$\lim_{\eta \to 0} \left[\log(x + i\eta) - \log(x - i\eta) \right] = -2\pi i$$

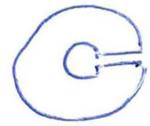
$$\Rightarrow \boxed{J(0, \infty) = -2\pi i \, I}$$

Example $\int_0^\infty \frac{dx}{(x+1)(x+2)(x+3)}$ Ans. $\ln 2 - \frac{1}{2} \ln 3$

① $\int_0^\infty \frac{\ln x \, dx}{(x^2+4)^2} = \frac{\pi}{32}(\ln 2 - 1)$



② $\int_0^\infty \frac{x^{-a}}{1+x} dx = \frac{\pi}{\sin a\pi}$ $0 < a < 1$



Laplace Transform

Some functions do not have convergent Fourier transforms.
e.g., $f(x) = 1$, $f(x) = x$, etc.

Sometimes, multiplication by e^{-cx} [$c \in \mathbb{R}$ and is greater than some constant depending on the function that was chosen] gets rid of the problem at $+\infty$. It however introduces additional problems at $-\infty$.

Consider the function $f(x)e^{-cx}\Theta(x)$ where $\Theta(x)$ is the Heaviside step function. $\Theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

This function may have a convergent Fourier transform.

$$F_1(k) = \int_{-\infty}^{\infty} f(x)\, e^{-cx}\,\Theta(x)\, e^{-ikx}\, dx$$

$$= \int_{0}^{\infty} f(x)\, e^{-(c+ik)x}\, dx \qquad \text{--- (1)}$$

Its inverse Fourier transform is,

$$f(x)\, e^{-cx}\,\Theta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(k)\, e^{ikx}\, dk \qquad \text{--- (2)}$$

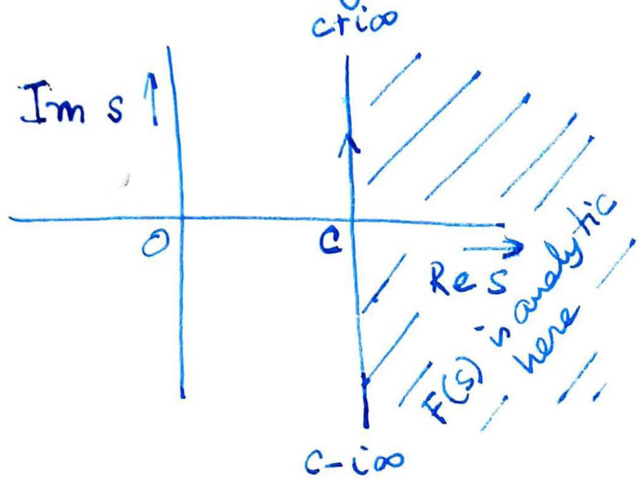
We make a change of variables: $s = c + ik$, and write $F(s) = F_1\left(\frac{s-c}{i}\right) = F_1(k)$ to get the Laplace transform of $f(x)$.

$$F(s) = \int_{0}^{\infty} f(x)\, e^{-sx}\, dx \qquad \text{--- (3)}$$

To get the inverse Laplace transform, we multiply Eq.(2) by e^{cx} and make a change of variables, $s = c + ik \Rightarrow k = \frac{s-c}{i}$

Thus, $$f(x)\,\Theta(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\, e^{sx}\, ds.$$

This integral is known as the Bromwich integral.

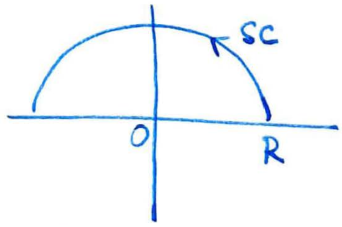


c is chosen so that $F(s)$ is analytic for $\mathrm{Re}\, s > c$.

Aside

Other forms of Jordan's lemma:

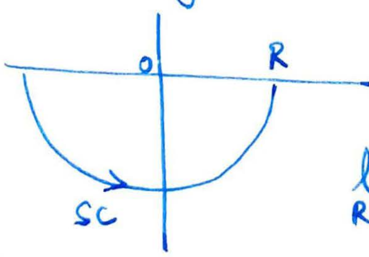
① Usual Form



If $\lim_{R\to\infty} f(Re^{i\theta}) = 0$ for $0 \leq \theta \leq \pi$,

$$\lim_{R\to\infty} \int_{SC} f(z)\, e^{iaz}\, dz = 0 \qquad a \in \mathbb{R} \text{ and } a > 0$$

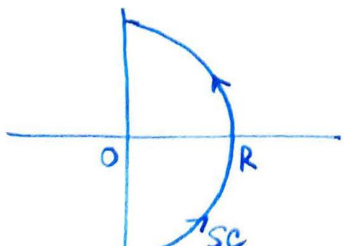
② [Change variables $z = -t$]



If $\lim_{R\to\infty} f(Re^{i\theta}) = 0$ for $\pi \leq \theta \leq 2\pi$

$$\lim_{R\to\infty} \int_{SC} f(z)\, e^{-iaz}\, dz = 0, \qquad a \in \mathbb{R},\ a > 0.$$

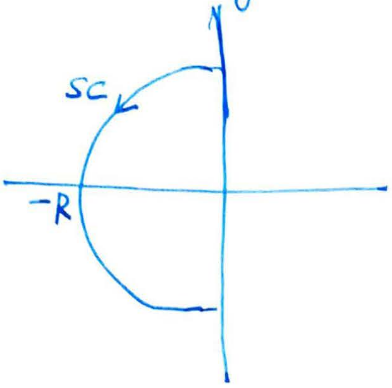
③ [Change variables $z = +it$]



If $\lim_{R\to\infty} f(Re^{i\theta}) = 0$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$\lim_{R\to\infty} \int_{SC} f(z)\, e^{-az}\, dz = 0, \qquad a \in \mathbb{R},\ a > 0.$$

④ [Change variables $z = -it$]



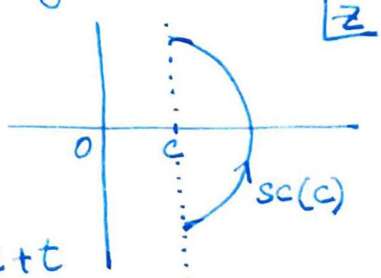
If $\lim_{R\to\infty} f(Re^{i\theta}) = 0$ for $\pi/2 \leq \theta \leq 3\pi/2$

$$\lim_{R\to\infty} \int_{sc} f(z)\, e^{az}\, dz = 0, \quad a \in \mathbb{R},\ a > 0$$

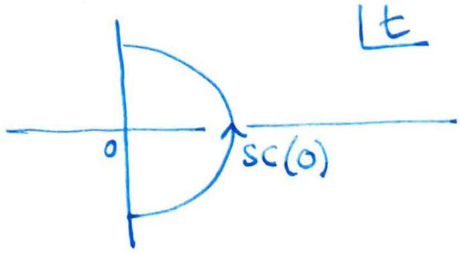
⑤ In all of the above results, you could shift the origin to an arbitrary point z_0 in the complex plane ($|z_0| < \infty$)

e.g. ~~from ③,~~ consider the integral,

$$\lim_{R\to\infty} \int_{sc(c)} f(z)\, e^{-az}\, dz$$



$$= \lim_{R\to\infty} \int_{sc(0)} f(c+t)\, e^{-a(c+t)}\, dt \qquad z = c+t$$



$$= e^{-ac} \lim_{R\to\infty} \int_{sc(0)} f(c+t)\, e^{-at}\, dt$$

$$= 0 \quad \text{if} \quad f(c+Re^{i\theta}) \to 0 \text{ as } R\to\infty \text{ for } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Note: The above results are also valid for any finite fraction of the semi-circle, e.g., a quarter circle.

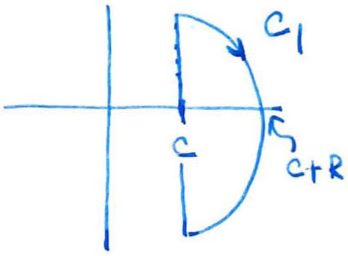
Example 1.

Find the function whose Laplace transform is $F(s) = \frac{1}{s}$.

$F(s)$ has a simple pole at the origin. We can therefore choose the constant c to be any positive real number.

We use the Bromwich integral,

$$f(x)\,\Theta(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\,e^{xs}\,ds \quad \cdots (1)$$



For $x<0$, we use the contour C_1.

$$\oint_{C_1} F(s)\,e^{xs}\,ds = 0 \quad [\because \text{ there are no singularities of } F(s) \text{ inside } C_1]$$

$$\Rightarrow \int_{c-i\infty}^{c+i\infty} F(s)\,e^{xs}\,ds + I_{SC_1} = 0$$

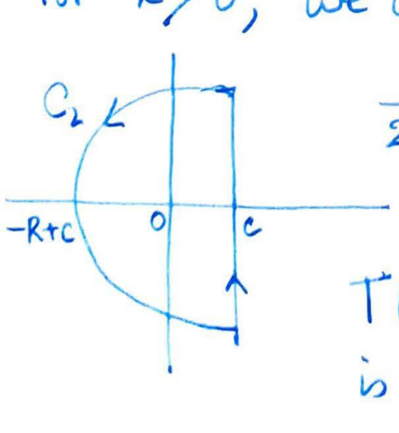
$I_{SC_1} = 0$ from Jordan's lemma since,

$$\lim_{R\to\infty} F(c+Re^{i\theta}) = \lim_{R\to\infty} \frac{1}{c+Re^{i\theta}} = 0$$

for $-\frac{\pi}{2} \leq \theta \leq \pi/2$

$\Rightarrow$ Both sides of Eq (1) are zero for $x<0$ (as expected).

For $x>0$, we close the contour on the opposite side



$$\frac{1}{2\pi i}\oint_{C_2} F(s)\,e^{xs}\,ds = \frac{1}{2\pi i} \times 2\pi i \times (\text{Residue at } s=0)$$

$$= \lim_{s\to 0} s \times \frac{e^{xs}}{s} = 1$$

The contribution from the semi-circular part is zero because of Jordan's lemma (Check.)

$\Rightarrow \quad f(x) = 1.$ [Check this result by Laplace transforming $f(x) = 1$.]

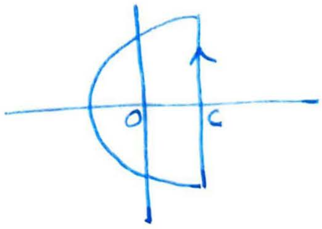
Example 2.

$$F(s) = \frac{1}{s - z}$$ where z is a complex number, $|z| < \infty$

Choose $c > \operatorname{Re} z$

Check the expected result for $x < 0$.

For $x > 0$ we use the same contour as in example 1, with $c > \operatorname{Re} z$.



From Jordan's lemma, the semi-circular part does not contribute,

$$\because \lim_{R \to \infty} \frac{1}{c + Re^{i\theta} - z} = 0 \quad \text{for } \pi/2 \leq \theta \leq 3\pi/2$$

$$\therefore f(x) = \frac{1}{2\pi i} \times 2\pi i \times (\text{Residue at } s = z)$$

$$= \lim_{s \to z} (s - z) \frac{e^{xs}}{s - z} = e^{xz}$$

[FILL IN THE MISSING STEPS AND MAKE SURE YOU UNDERSTAND EVERYTHING.]

(a) For $z = \lambda$, ($\lambda \in \mathbb{R}$), inverse Laplace transform of $\frac{1}{s - \lambda}$ is $e^{\lambda x}$

(b) $z = i\lambda$. Inverse Laplace transform of $\frac{1}{s - i\lambda}$ is $e^{i\lambda x}$.

Since Laplace transform is linear,

$$\mathcal{L}^{-1}\left[\operatorname{Re}\frac{1}{s-i\lambda}\right] = \operatorname{Re} e^{i\lambda x}$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s}{s^2+\lambda^2}\right] = \cos\lambda x$$

and $$\mathcal{L}^{-1}\left[\operatorname{Im}\frac{1}{s-i\lambda}\right] = \operatorname{Im} e^{i\lambda x}$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{\lambda}{s^2+\lambda^2}\right] = \sin\lambda x$$

Find the inverse Laplace transform of $\frac{1}{s^n}$, $n \in \mathbb{Z}, n>0$.