Complex Variables

Saurish Chakrabarty

Department of Physics, Acharya Prafulla Chandra College

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Syllabus

PG (APC College): 1. Complex numbers 2. Triangular inequalities 3. Schwarz inequality 4. Function of a complex variable (a) Single and Multiple-valued function (b) Limits and Continuity 5. Differentiation (a) Cauchy-Riemann equations and their applications (b) Analytic function (c) Harmonic function 6. Complex integrals (a) Cauchy's theorem (elementary proof only) (b) Converse of Cauchy's theorem (c) Cauchy's Integral Formula and its corollaries 7. Series (a) Taylor (b) Laurent 8. Classification of singularities 9. Branch point and branch cut 10. Residue theorem and evaluation of some typical real integrals using this theorem.

UG (WBSU): 1. Euler's formula 2. De Moivre's theorem 3. Roots of Complex Numbers 4. Functions of Complex Variables. 5. Analyticity and Cauchy-Riemann Conditions. 6. Examples of analytic functions. 7. Singular points: poles and branch points, order of singularity, branch cuts. Laurent and 8. Taylor's expansion 9. Integration of a function of a complex variable. (a) Cauchy's inequality (b) Cauchy's integral formula (c) Simply and multiply connected regions (d) Residues and Residue Theorem (e) Application in solving definite and improper integrals

1 Recap of things from Class XI-XII

A complex number is an ordered pair of real numbers, represented as $z = (x, y) = x + iy$. In polar representation $z = re^{i\theta}$, r and θ being respectively the *modulus* and *argument* of z. The specification of a complex number requires two independent pieces of information – (x, y) or (r, θ) or (z, z^*), where z^* is the complex conjugate of z, $z^* = x - iy = re^{-i\theta}$.

1.1 n^{th} roots of a complex number

The n^{th} roots of a complex number z_0 are the solutions of the equation $z^n = z_0$. Writing z_0 in polar form as $r_0 e^{i\theta_0}$ and inserting a factor of $e^{2\pi i}$ into the equation, we see that the n distinct roots can be written as $\sqrt[n]{r_0}\ e^{i\left(\frac{\theta_0}{n}+\frac{2\pi k}{n}\right)}$, where $k \in \{0, 1, 2, \ldots, n-1\}$, where $\sqrt[n]{\cdot}$ denotes the real positive n^{th} root of a positive real number. The n^{th} of unity are easily obtained from this formula as below.

2	3	4	5	6
1 $e^{\pi i}$	1 $e^{2\pi i/3}$ $e^{4\pi i/3}$	1 $e^{\pi i/2}$ $e^{\pi i}$ $e^{3\pi i/2}$	1 $e^{2\pi i/5}$ $e^{4\pi i/5}$ $e^{6\pi i/5}$ $e^{8\pi i/5}$	1 $e^{\pi i/3}$ $e^{2\pi i/3}$ $e^{\pi i}$ $e^{4\pi i/3}$ $e^{5\pi i/3}$
± 1	$1, \frac{-1\pm i\sqrt{3}}{2}$	$\pm 1, \pm i$	$\pm 1, \cos\dfrac{2\pi}{5} \pm i\sin\dfrac{2\pi}{5}$, $\cos\frac{4\pi}{5} \pm i\sin\frac{4\pi}{5}$	$\pm 1, \frac{\pm 1\pm i\sqrt{3}}{2}$

1.2 Equations of curves in the complex plane

Circle: $|z - z_0| = r$ (Circle of radius r centered at z_0).
Ellipse: $|z - z_1| + |z - z_2| = 2a$ (Ellipse with semimajor axis a having foci at z_1 and z_2).
Hyperbola: $\Big||z - z_1| - |z - z_2|\Big| = 2a$ (Hyperbola with semimajor axis a having foci at z_1 and z_2).
Straight line: $|z - z_1| = |z - z_2|$

Note that an equation in terms of (x, y) may be translated to one involving z and z^* by substituting $x = \dfrac{z + z^*}{2}$ and $y = \dfrac{z - z^*}{2i}$.

1.3 Inequalities

$$|z|^2 = |\Re z|^2 + |\Im z|^2$$
$$\Rightarrow |z| \geq |\Re z| \geq \Re z, \text{ and, } |z| \geq |\Im z| \geq \Im z. \quad (1)$$

$$\textit{Triangle Inequality:} \quad |z_1 + z_2| \leq |z_1| + |z_2| \quad (2)$$

2 Riemann sphere (stereographic projection)

All points on the infinite complex plane may be mapped to points on the surface of a unit sphere. Such a sphere is called the Riemann sphere. There are many such mappings. One of them is described below.
Let the complex plane be the xy-plane of a three dimensional Cartesian coordinate system. Consider the unit sphere centered at the origin as the Riemann sphere on which all points in the complex plane would be mapped. For every point $z = x + iy \equiv (x, y)$ on the complex plane a line is drawn to the "north pole", (0,0,1). The point where this line intersects the surface of the sphere is the mapping of z on the Riemann sphere. it is obvious that all points for which $z > 1$ get mapped to the northern hemisphere and those for which $z < 1$ get mapped to the southern hemisphere. The circle $z = 1$ becomes the equator and the south and north poles correspond to the origin and the point at infinity respectively. Through this mapping, it is easy to appreciate that all ways of running off to infinity takes us to the same point which is not true for real variables.
Check that circles and straight lines on the complex plane get mapped to circles on the surface of the Riemann sphere.

3 Function of a complex variable

A function $f(z)$ of a complex variable can be thought of as an ordered pair of two real-valued functions of x and y, i.e., $f(z) \equiv u(x, y) + i\ v(x, y)$. Looking simultaneously at the plots of the surfaces, $u(x, y)$ and $v(x, y)$, is a way to visualize the function $f(z)$. An alternate way to visualize the same function is as a mapping from the complex z-plane to the complex w-plane. The w-plane contains points which denote the values taken by the function $f(z)$.

An example of a function is $f(z) = z^2$. This maps the points on the upper half of the z-plane to the entire w-plane. It should be easy to see that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

The *limit* as z approaches z_0, denoted by $\lim\limits_{z \to z_0}$, is the continuous sequence of points in the complex plane approaching some given complex number z_0 *along an arbitrary path.*

A function is said to be *continuous* at z_0 if $f(z_0)$ exists and $\lim\limits_{z \to z_0} f(z) = f(z_0)$.
An example of a function is $f(z) = z^2$. This maps the upper half plane to the entire complex plane does the same for lower half.
For the function, $f(z) = z^{1/2}$, the situation is trickier. This function appears multi-valued, except at $z = 0$ and at $z = \infty$. It is made single valued by expanding our domain to two sheets which are copies of the complex plane. These sheets are called *Riemann sheets* and the set of sheets for each function is called the *Riemann surface.* If a point z_0 on the first sheet maps to the point w_0, where $w_0^2 = z_0$, then the point on the second sheet located at the same value z_0 maps to $-w_0$. The two sheets are connected along a line (or a non-intersecting curve) which goes from zero to the point at infinity in any direction as per our convenience. These two points are called *branch points* for the function $z^{1/2}$ and the line is called the *branch cut.* For a point to be a branch point, encircling it by continuously changing the argument of z, must take us to a different sheet (different value of the function). In simple words, if the residents of sheet one follow the convention that $4^{1/2} = +2$, those on sheet two follow the convention that $4^{1/2} = -2$. If we encircle the origin once (and therefore cross the branch cut), then we automatically change sheets and therefore switch our convention. At every point on the Riemann surface where the function takes a finite non-zero value, it remains single valued and continuous.

Examples of other functions requiring multiple Riemann sheets

- $z^{1/5}$: Five sheets. Branch points at 0 and ∞.
- $\ln z$: Infinite number of sheets. Branch points at 0 and ∞. $\ln z = \ln(re^{i(\theta+2n\pi)}) = \ln r + i(\theta + 2n\pi)$, $n \in \mathbb{Z}$, $\ln r \in \mathbb{R}$ and $-\pi < \theta \leq \pi$.
- i^z: Infinite number of sheets. No branch point (Riemann sheet not changed on encircling any point). $i^z = e^{i(\pi/2+2n\pi)z} = e^{i(\pi/2+2n\pi)r(\cos\theta+i\sin\theta)} =$.

4 Analyticity

4.1 Cauchy-Riemann conditions

5 Complex Integrals

5.1 Cauchy's Theorem

5.2 Cauchy's Integral Theorem

5.3 Taylor Series

5.4 Laurent Series

5.5 Residue Theorem

5.6 Cauchy's Principal Value

5.7 Jordan's Lemma

5.8 Laplace Transform

Reference

- R. V. Churchill and J. W. Brown, "Complex Variables and Applications"
- V. Balakrishnan, "Mathematical Physics"
- J. Mathews and R. L. Walker, "Mathematical Methods of Physics"
