Lagrangian formulation of Special Relativity

[Note: We will only discuss a non covariant formulation. You can read about covariant formulations on your own.]

A relativistic particle with rest mass m, moving under the influence of a potential energy V, can be described using the Lagrangian,

$$L = -mc^2\sqrt{1-\beta^2} - V$$

In the non relativistic limit, $\sqrt{1-\beta^2} \sim -\frac{1}{2}\frac{v^2}{c^2} \Rightarrow L = \frac{1}{2}mv^2 - V + \text{constant}$ (as expected).

Generalization to many particles: $L = -\sum_s m_s c^2\sqrt{1-\beta_s^2} - V.$

Equation of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) = \frac{\partial L}{\partial x_i}$$

$$\frac{\partial L}{\partial v_i} = \frac{-mc^2}{2\sqrt{1-\beta^2}}\left(\frac{-2v_i}{c^2}\right) = \gamma m v_i = p_i \quad , \quad \frac{\partial L}{\partial x_i} = -\frac{\partial V}{\partial x_i} = F_i$$

$$\Rightarrow \frac{d\vec{p}}{dt} = \vec{F}$$

If L doesn't depend on time explicitly, then, the energy is conserved.

$\Rightarrow h = p\dot{q} - L$, considering a single particle in one dimension.

$$= \gamma m v^2 + mc^2\sqrt{1-\beta^2} + V$$

$$= \gamma mc^2\left(\frac{v^2}{c^2} + 1 - \beta^2\right) + V$$

$$= \gamma mc^2 + V = \text{Total energy.}$$

Velocity dependent potential : $V = q\phi - q\vec{A}\cdot\vec{v}$

Canonical momentum $p_i = \dfrac{\partial L}{\partial v_i} = \gamma m v_i + q A_i$

One dimensional motion under the influence of a constant force

e.g. under the influence of earth's gravity near the earth's surface.

Let the force per unit mass be 'a' $\Rightarrow V = -max$

Let at $t=0$, $x=0$ and $\dot{x}=0$.

[Non-relativistic answer: $x = \frac{1}{2}at^2$ (parabola in x-t plane).]

Using $L = -mc^2\sqrt{1-\beta^2} + max$, we get,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) = \frac{\partial L}{\partial x} \Rightarrow \frac{d}{dt}(\gamma m v) = ma$$

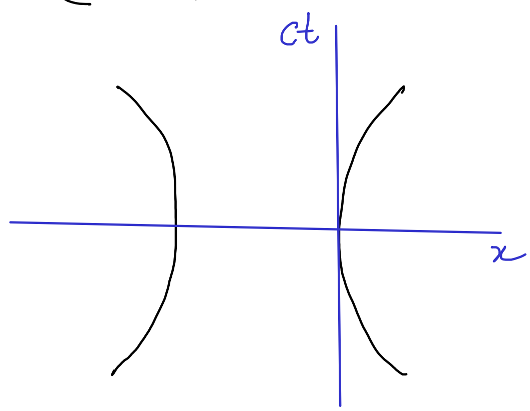
$$\Rightarrow \gamma v = at \quad \because \text{ at } t=0,\ v=0$$

$$\Rightarrow 1-\beta^2 = \left(\frac{v}{at}\right)^2 \Rightarrow v^2\left(\frac{1}{(at)^2} + \frac{1}{c^2}\right) = 1$$

$$\Rightarrow \frac{dx}{dt} = \frac{atc}{\sqrt{(at)^2 + c^2}} = \frac{ct}{\sqrt{t^2 + (c/a)^2}} \quad ---(1)$$

$$\Rightarrow x = c\sqrt{t^2 + (c/a)^2}\,\Big|_0^t = c\sqrt{t^2+(c/a)^2} - c^2/a$$

$$\Rightarrow (x + c^2/a)^2 - c^2t^2 = c^4/a^2 \quad \rightarrow \text{Hyperbola}$$

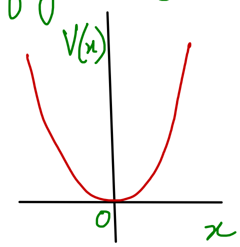


[To get the non relativistic answer, assume $c^2 \gg at^2$ in Eq. (1)

$\Rightarrow \dfrac{dx}{dt} = at \Rightarrow x = \frac{1}{2}at^2$]

Relativistic Harmonic Oscillator

Let us consider a general symmetric potential energy $V(x)$ with a minimum at the origin.



$$L = -mc^2\sqrt{1-\beta^2} - V(x)$$

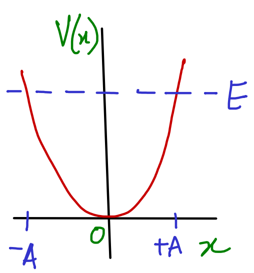
No explicit time dependence in $L \Rightarrow$ Conserved energy.

$$E = \gamma mc^2 + V(x)$$

$$\Rightarrow \gamma = \frac{E-V}{mc^2} \Rightarrow 1-\beta^2 = \left(\frac{mc^2}{E-V(x)}\right)^2 \Rightarrow \beta^2 = 1-\left(\frac{mc^2}{E-V(x)}\right)^2$$

$$\Rightarrow \frac{dx}{dt} = v = c\sqrt{1-\left(\frac{mc^2}{E-V(x)}\right)^2}$$

$$\Rightarrow \frac{dx}{c\sqrt{1-\left(\frac{mc^2}{E-V(x)}\right)^2}} = dt$$



Turning points: Kinetic Energy $= 0 \Rightarrow E - mc^2 - V(x) = 0$

$\Rightarrow x = \pm A \quad \Rightarrow E - mc^2 = V(\pm A)$

$\therefore$ Time period, $$T = \frac{4}{c}\int_0^A \frac{dx}{\sqrt{1-\left(\frac{mc^2}{E-V(x)}\right)^2}}$$

In the harmonic case, i.e., $V(x) = \frac{1}{2}kx^2$, we will find the leading relativistic correction to the nonrelativistic answer, $T = 2\pi\sqrt{\frac{m}{k}}$.

$$\sqrt{1-\left(\frac{mc^2}{E-V(x)}\right)^2} = \sqrt{\frac{(E-V(x))^2-(mc^2)^2}{(E-V(x))^2}} = \frac{\sqrt{E-V(x)-mc^2}\sqrt{E-V(x)+mc^2}}{E-V(x)}$$

$\therefore E - mc^2 = V(\pm A) = \frac{1}{2}kA^2$

$$\Rightarrow \sqrt{1-\left(\frac{mc^2}{E-V(x)}\right)^2} = \frac{\sqrt{k(A^2-x^2)/2}\;\sqrt{k(A^2-x^2)/2+2mc^2}}{mc^2+\frac{1}{2}k(A^2-x^2)}$$

Assuming $mc^2 \gg \frac{1}{2}kA^2 \geq \frac{1}{2}kx^2$

$$T = \frac{4}{c}\int_0^A mc^2\left(1+\frac{k(A^2-x^2)}{2mc^2}\right)\frac{1}{\sqrt{k(A^2-x^2)/2}}\;\frac{1}{\sqrt{2mc^2}}\left(1+\frac{k(A^2-x^2)}{4mc^2}\right)^{-1/2}dx$$

$$\simeq 4\sqrt{\frac{m}{k}}\int_0^A \frac{dx}{\sqrt{A^2-x^2}} + 4\sqrt{\frac{m}{k}}\,\frac{k}{mc^2}\int_0^A \sqrt{A^2-x^2}\left(\frac{1}{2}-\frac{1}{8}\right)dx$$

$$= 4\sqrt{\frac{m}{k}}\int_0^{\pi/2} d\theta + 4\sqrt{\frac{m}{k}}\,\frac{kA^2}{mc^2}\,\frac{3}{8}\int_0^{\pi/2}\cos^2\theta\, d\theta$$

$x = A\sin\theta$

$dx = A\cos\theta\, d\theta$

$$= 2\pi\sqrt{\frac{m}{k}} + \frac{3}{2}\sqrt{\frac{m}{k}}\,\frac{kA^2}{mc^2}\,\frac{1}{2}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$$

$$= 2\pi\sqrt{\frac{m}{k}} + \frac{3}{4}\sqrt{\frac{m}{k}}\,\frac{kA^2}{mc^2}\,\frac{\pi}{2}$$

$$= 2\pi\sqrt{\frac{m}{k}}\left(1+\frac{3}{16}\,\frac{kA^2}{mc^2}\right)$$

[There is a typo in Goldstein's book here.]

↳ The time period depends on the amplitude of oscillation!