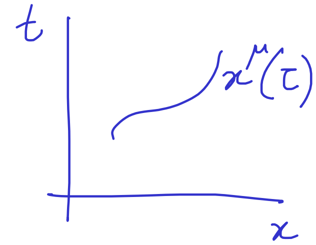
# FOUR VECTORS

**4-position:**

$x^\mu$ : $\quad x^0 = ct, \quad (x^1, x^2, x^3) = \vec{r}$

Timelike curves are parametrized by the proper time, $\tau$.




**4-velocity:**

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau}$$

$$= \gamma \frac{dx^\mu}{dt}$$

$$= (\gamma c, \gamma \vec{v})$$

$dt = \gamma d\tau$

$\Rightarrow \frac{dt}{d\tau} = \gamma$

**4-momentum:**

$$p^\mu = m u^\mu = (\gamma m c, \gamma m \vec{v})$$

$m =$ rest mass

$$= (E/c, \vec{p})$$

$E = \gamma m c^2, \quad \vec{p} = \gamma m \vec{v}$

**4-acceleration:**

$$a^\mu = \frac{du^\mu}{d\tau} = \gamma \frac{du^\mu}{dt} = \gamma\left(c\dot{\gamma}, \dot{\gamma}\vec{v} + \gamma\vec{a}\right), \quad \dot{\gamma} \equiv \frac{d\gamma}{dt}, \vec{a} \equiv \frac{d\vec{v}}{dt}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \Rightarrow \dot{\gamma} = \frac{1}{2(1-\beta^2)^{3/2}}\frac{d\beta^2}{dt} = \frac{\gamma^3}{2}\frac{d}{dt}(\vec{v}\cdot\vec{v}/c^2)$$

$$= \frac{\gamma^3}{c^2}\vec{v}\cdot\vec{a}$$

$$\therefore \quad a^\mu = \left(\frac{\gamma^4}{c}\vec{v}\cdot\vec{a}, \; \frac{\gamma^4}{c^2}(\vec{v}\cdot\vec{a})\vec{v} + \gamma^2\vec{a}\right)$$

Proper 4-acceleration (measured in frame where $\vec{v} = 0$)

$$a^\mu_{\text{proper}} = (0, \vec{a})$$

**4-force:**

$$K^\mu = \frac{dp^\mu}{d\tau} = \gamma\frac{dp^\mu}{dt} = \left(\frac{\gamma}{c}\frac{dE}{dt}, \gamma\frac{d\vec{p}}{dt}\right)$$

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$\Rightarrow K^\mu = \left(\frac{\gamma}{c}\frac{dE}{dt}, \gamma\vec{F}\right)$$

$$K^\mu_{proper} = \left(0, \vec{F}\right), \qquad \vec{v}=0, \quad E = mc^2 = constant$$

## Scalar product in Minkowski coordinate system

Metric tensor in Minkowski space, $g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$$v^\beta g_{\alpha\beta} = v_\alpha \Rightarrow u.v = u^\alpha v_\alpha$$

$$u.v \equiv u^\alpha v^\beta g_{\alpha\beta} = u^0 v^0 - \vec{u}.\vec{v}$$

$$\Rightarrow u.u = u^\mu u_\mu = u^0 u^0 - \vec{u}^2 \begin{cases} >0 \Rightarrow \text{Timelike} \\ <0 \Rightarrow \text{Spacelike} \\ =0 \Rightarrow \text{Null / Lightlike} \end{cases}$$

For 4-velocity, $u.u = \gamma^2 c^2 - \gamma^2 \vec{v}^2 = \gamma^2 c^2 (1/\gamma^2) = c^2$

$\therefore$ The 4-velocity is timelike.

For 4-acceleration, if we look at the proper 4-acceleration,

$$a_{proper} \cdot a_{proper} = -\vec{a}^2 \leq 0$$

$\Rightarrow$ The 4-acceleration is spacelike (or null if $\vec{a}=0$).

For 4-momentum, $p.p = m^2\, u.u = m^2c^2 > 0$ (timelike)

For 4-force, again looking at the proper 4-force,

$$K_{proper} \cdot K_{proper} = -\vec{F}^2 \quad \text{(spacelike)}$$

$$a.v = \left(\frac{\gamma^4}{c}\vec{v}.\vec{a}, \frac{\gamma^4}{c^2}(\vec{v}.\vec{a})\vec{v} + \gamma^2\vec{a}\right) \cdot \left(\gamma c, \gamma\vec{v}\right)$$

$$= \gamma^5 \vec{v}.\vec{a} - \gamma^5 (\vec{v}.\vec{a}) v^2/c^2 - \gamma^3 \vec{a}.\vec{v}$$

$$= \gamma^5 \vec{v}.\vec{a}\,(1/\gamma^2) - \gamma^3 \vec{v}.\vec{a} = 0$$

[The scalar product of the 4-velocity and the 4-acceleration is always zero.]

# Relativistic Kinematics

– Multiparticle collisions

Center of Momentum (COM) frame: Reference frame where the total 3-momentum of all the particles adds up to zero.

Total 4-momentum, $p^\mu = \sum_s p_s^\mu$ s labels particles

$$p^\mu p_\mu = \left(\sum_s \gamma_s m_s c\right)^2 - \left(\sum_s \gamma_s m_s \vec{v}_s\right)^2$$

$$= \sum_{rs} \gamma_r \gamma_s m_r m_s c^2 - \sum_{rs} \gamma_r \gamma_s m_r m_s \vec{v}_r \cdot \vec{v}_s$$

$$= \sum_s \gamma_s^2 m_s^2 \underbrace{(c^2 - v_s^2)}_{c^2/\gamma_s^2} + \sum_{r\neq s} \gamma_r \gamma_s m_r m_s \left(c^2 - \vec{v}_r \cdot \vec{v}_s\right)$$

$$= \sum_s m_s^2 c^2 + \sum_{r\neq s} \gamma_r \gamma_s m_r m_s \left(c^2 - \vec{v}_r \cdot \vec{v}_s\right) \geqslant 0 \quad \star$$

$$= \sum_s m_s^2 c^2 + \sum_{r\neq s} p_r^\mu p_{s\mu}$$

$\star \Rightarrow$ The total 4-momentum is timelike $\Rightarrow$ there exists a frame in which the total 3-momentum vanishes — the COM frame.

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Consider a reaction in which two particles collide and produce a set of particles. Consider the 2nd particle to be at rest (stationary target).

$r = 1, 2 \to$ Colliding particles; $r = 3, 4, \ldots \to$ Produced particles

$p^\mu p_\mu$ is invariant in all inertial frames and is conserved in the reaction.

COM frame $\to$ primed frame.

For initial particles,

In COM frame, $p^{\mu\prime} = (E'/c, 0, 0, 0)$

In the lab frame, $p^\mu p_\mu = (m_1^2+m_2^2)c^2 + 2p_1^\mu p_{2\mu}$

$$= (m_1^2+m_2^2)c^2 + 2(E_1E_2/c^2 - \vec{p}_1\cdot\vec{p}_2)$$

$$= (m_1^2+m_2^2)c^2 + 2m_2E_1$$

$$= (m_1^2+m_2^2)c^2 + 2m_2(T_1+m_1c^2)$$

$$= (m_1+m_2)^2c^2 + 2m_2T_1$$

$$\therefore \quad E'^2/c^2 = (m_1+m_2)^2c^2 + 2m_2T_1$$

$$\Rightarrow \quad E'^2 = (m_1+m_2)^2c^4 + 2m_2c^2T_1$$

The treshold condition for the reaction corresponds to the situation where the final particles are produced at rest in the COM frame.

At threshold, after the reaction, in the COM frame,

$$p^\mu p_\mu = \left(\sum_r m_r c^2\right)^2/c^2 = c^2\left(\sum_r m_r\right)^2$$

$\sum_r \equiv \sum_{r=3}^{n+2}$ if $n$ particles are produced

$$\therefore \quad (m_1+m_2)^2c^2 + 2m_2T_1^{th} = c^2\left(\sum_r m_r\right)^2$$

$$\frac{T_1^{th}}{m_1c^2} = \frac{\left(\sum_r m_r\right)^2 - (m_1+m_2)^2}{2m_1m_2}$$

Let $\quad Q = \left[\sum_r m_r - (m_1+m_2)\right]c^2$

$$\therefore \left(\sum_r m_r\right)^2 - (m_1+m_2)^2 = (Q/c^2)\left[\sum_r m_r + (m_1+m_2)\right]$$

$$= Q\left[Q/c^2 + 2(m_1+m_2)\right]/c^2$$

$$\Rightarrow \quad \frac{T_1^{th}}{m_1c^2} = \frac{Q^2 + 2Q(m_1+m_2)c^2}{2m_1m_2c^4}$$

Consider the reaction $p + n \longrightarrow p + n + p + \bar{p}$

Masses of $p, n, \bar{p} \approx 938\ \text{MeV}/c^2 \equiv m$

$\Rightarrow$ Threshold K.E. = ?

$$Q \approx 2mc^2 \quad \Rightarrow T_1 \approx \frac{4 + 2\times2\times2}{2} mc^2 = 6mc^2 = 5.6\,\text{GeV}$$

$$\Rightarrow T_1 = 3Q$$

If we were in the COM frame, all the K.E. would be available for the $p\bar{p}$ production and each incident particle would need a kinetic energy of 938 MeV.

# Lagrangian formulation of Special Relativity

[Note: We will only discuss a non covariant formulation. You can read about covariant formullations on your own.]

A relativistic particle with rest mass $m$, moving under the influence of a potential energy $V$, can be described using the Lagrangian,

$$L = -mc^2\sqrt{1-\beta^2} - V$$

In the non relativistic limit, $\sqrt{1-\beta^2} \sim -\frac{1}{2}\frac{v^2}{c^2} \Rightarrow L = \frac{1}{2}mv^2 - V + \text{constant}$ (as expected).

Generalization to many particles: $L = -\sum_s m_s c^2\sqrt{1-\beta_s^2} - V.$

Equation of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) = \frac{\partial L}{\partial x_i}$$

$$\frac{\partial L}{\partial v_i} = \frac{-mc^2}{2\sqrt{1-\beta^2}}\left(\frac{-2v_i}{c^2}\right) = \gamma m v_i = p_i \quad , \quad \frac{\partial L}{\partial x_i} = -\frac{\partial V}{\partial x_i} = F_i$$

$$\Rightarrow \quad \frac{d\vec{p}}{dt} = \vec{F}$$

If $L$ doesn't depend on time explicitly, then, the energy is conserved.

$\Rightarrow h = p\dot{q} - L$ , considering a single particle in one dimension.

$$= \gamma m v^2 + mc^2\sqrt{1-\beta^2} + V$$

$$= \gamma mc^2\left(\frac{v^2}{c^2} + 1 - \beta^2\right) + V$$

$$= \gamma mc^2 + V \quad = \text{Total energy.}$$

## One dimensional motion under the influence of a constant force

e.g. under the influence of earth's gravity near the earth's surface.

Let the force per unit mass be '$a$' $\Rightarrow V = -max$

Let at $t=0$, $x=0$ and $\dot{x}=0$.

[Non-relativistic answer: $x = \frac{1}{2}at^2$ (parabola in $x$-$t$ plane).]

Using $L = -mc^2\sqrt{1-\beta^2} + max$, we get,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) = \frac{\partial L}{\partial x} \Rightarrow \frac{d}{dt}(\gamma m v) = ma$$

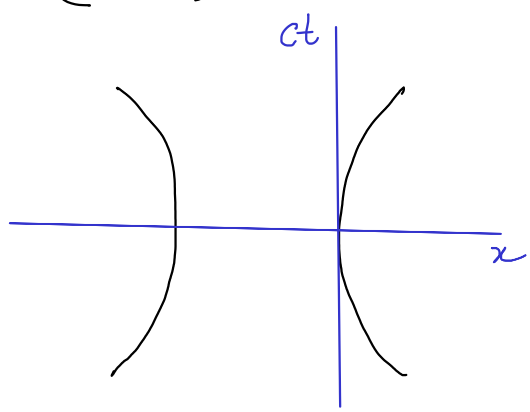
$$\Rightarrow \gamma v = at \quad \because \text{ at } t=0,\ v=0$$

$$\Rightarrow 1-\beta^2 = \left(\frac{v}{at}\right)^2 \Rightarrow v^2\left(\frac{1}{(at)^2} + \frac{1}{c^2}\right) = 1$$

$$\Rightarrow \frac{dx}{dt} = \frac{atc}{\sqrt{(at)^2+c^2}} = \frac{ct}{\sqrt{t^2+(c/a)^2}} \quad \text{---(1)}$$

$$\Rightarrow x = c\sqrt{t^2+(c/a)^2}\,\Bigg|_0^t = c\sqrt{t^2+(c/a)^2} - c^2/a$$

$$\Rightarrow (x + c^2/a)^2 - c^2t^2 = c^4/a^2 \longrightarrow \text{Hyperbola}$$




[To get the non relativistic answer, assume $c^2 \gg at^2$ in Eq. (1)

$\Rightarrow \frac{dx}{dt} = at \Rightarrow x = \frac{1}{2}at^2$]