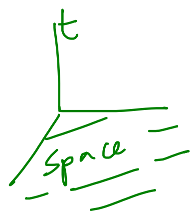
Special Relativity

Newton — Equivalent inertial reference frames

Maxwell — Universal speed of light $\Rightarrow$ Special inertial frame

Einstein — Special relativity — ① Laws of physics ② Speed of light } same for all inertial observers

Spacetime



Point in spacetime $\rightarrow$ Event

Separation between events $= \Delta s$ Interval in Minkowski space

$$(\Delta s)^2 = c^2 (\Delta t)^2 - (\Delta \vec{r})^2$$

Infinitesimal separation : $(ds)^2 = c^2 (dt)^2 - (d\vec{r})^2$

For light, $ds = 0$, i.e., $\dfrac{(d\vec{r})^2}{(dt)^2} = c^2$

For actual bodies, $\dfrac{(d\vec{r})^2}{(dt)^2} < c^2 \Rightarrow (ds)^2 > 0$ "Timelike interval"

$ds = 0$ "Light like" or "null" interval

$(ds)^2 < 0$ "Spacelike" interval

Objects moving on timelike intervals $\rightarrow$ Tardyons

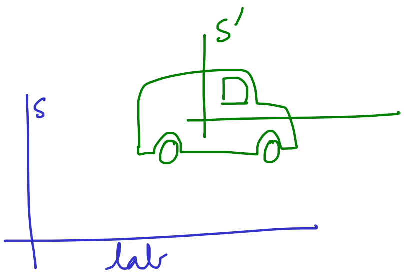
Hypothetical objects moving on spacelike intervals $\rightarrow$ Tachyons

For special relativity, 4D Euclidean space can be used, after redefining time interval as $dt_e = i\,dt$.

All inertial observers measure intervals of the same length in Minkowski space.

$$(ds')^2 = (ds)^2 \qquad \text{INVARIANT SPACETIME INTERVAL}$$

Time measured by clock at rest w.r.t. moving body $\rightarrow$ Proper time



$S: (t, x, y, z)$

$S': (\tau, x', y', z')$

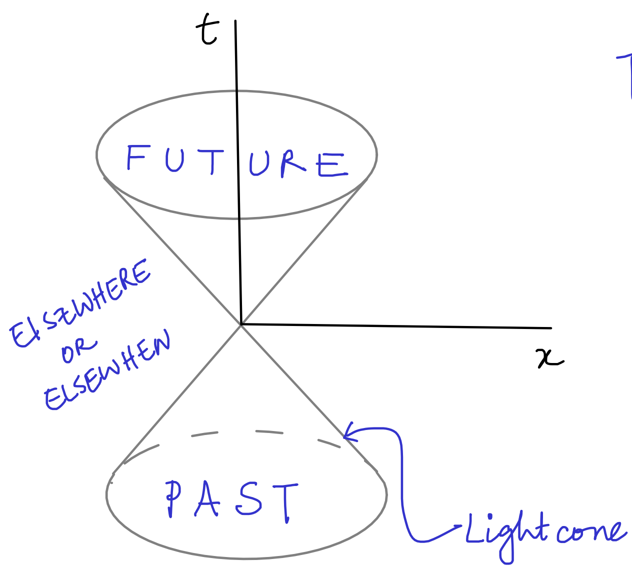
τ → proper time

$$c^2(d\tau)^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$
$$= c^2(dt)^2 - v^2(dt)^2$$
$$\Rightarrow (dt)^2 = \frac{(d\tau)^2}{1 - v^2/c^2}$$

$$\Rightarrow dt = \gamma\, d\tau, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$ "Time dilation"

Length measured in S' frame for object at rest in S' = Δl

Length measured in S frame = $\Delta l / \gamma$ "Length contraction"



The light cone separates spacetime into three regions

- Future
- Past
- Elsewhere / Elsewhen

If the origin of spacetime corresponds to an event A, then the future contains events, B, for which $t_B > t_A$ and $(\Delta s)^2 > 0$. Event B occurs after event A in all inertial reference frames.

The past contains events C for which $t_C < t_A$. Again, C occurs before A in all inertial reference frames.

A and B (and, A and C) are causally connected. There exists a reference frame in which A and B occur at the same point in space, but at different times. — Timelike separation

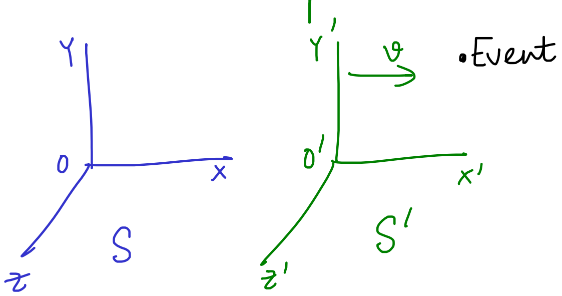
Outside the lightcone, time ordering of events is arbitrary. For an event $\mathcal{D}$ outside the lightcone,

there exists a reference frame in which the events $\mathcal{A}$ and $\mathcal{D}$ occur at the same time, but at different points in space.
— Space like separation.

Lorentz transformation

Simplest set of transformations that preserve the invariance of $(ds)^2$.
↳ ① Linear
② As $v \to 0$, $L \to \mathbb{1}$.

Consider frames S and S', whose origins coincide when $t = t' = 0$ and whose axes are parallel.



Point of view of S: $OO' = vt$
$\therefore x - vt = x'/\gamma$

Point of view of S': $OO' = vt'$
$\therefore vt' + x' = x/\gamma$

$$\therefore \quad x' = \gamma(x - vt) = \gamma(x - \beta ct) \quad \text{---}(1)$$

$$\beta ct' + x' = x/\gamma \Rightarrow ct' = x/_{\gamma\beta} - x'/_{\beta} = x/_{\gamma\beta} - \gamma(x - \beta ct)/_{\beta}$$

$$\Rightarrow ct' = \gamma\left[ct + \frac{1}{\beta}\left(\frac{1}{\gamma^2} - 1\right)x\right] = \gamma(ct - \beta x) \quad \text{----}(2)$$

$$\therefore \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$L = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ is a Lorentz transformation

that corresponds to a pure boost along the x-direction.

Pure boost with $\vec{v}$ in arbitrary direction keeping axes parallel

In Eq. (2), we must replace βx by $\vec{\beta}\cdot\vec{r}$, where $\vec{\beta} = \vec{v}/c$,

i.e. $$ct' = \gamma\left(ct - \vec{\beta}\cdot\vec{r}\right)$$

When the boost is along the x-direction, $\Delta\vec{r} = \vec{r}\,' - \vec{r}$ is along the x-direction. When it is along a different direction $\Delta\vec{r}$ must be along that direction.

In the former case,

$$\Delta\vec{r} = \left((\gamma-1)x - \gamma\beta ct, 0, 0\right)$$

$$= (\gamma-1)(x,0,0) - \gamma ct\vec{\beta} \qquad \because \text{ In this case, } \vec{\beta} = (\beta,0,0)$$

$$= \frac{(\gamma-1)(\vec{\beta}\cdot\vec{r})\vec{\beta}}{\beta^2} - \gamma ct\vec{\beta}$$

Thus, in general, $$\vec{r}\,' = \vec{r} + \frac{(\gamma-1)(\vec{\beta}\cdot\vec{r})}{\beta^2}\vec{\beta} - \gamma ct\vec{\beta}$$

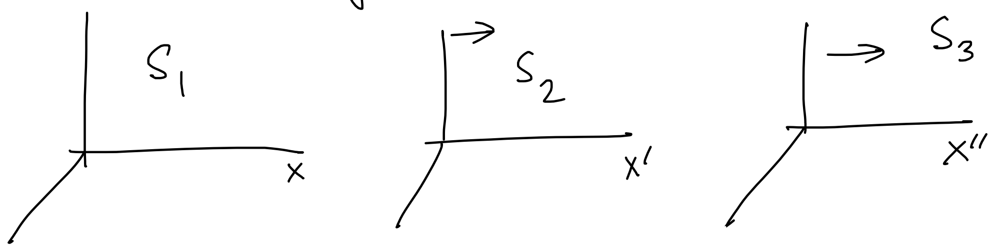
$$\Rightarrow \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = L \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$ where $L_{00} = \gamma$, $L_{0i} = L_{i0} = -\gamma\beta_i$

$$L_{ij} = \delta_{ij} + \frac{\gamma-1}{\beta^2}\beta_i\beta_j$$

A pure boost is represented by a **symmetric** matrix.

[Note: If 4-origins don't match, $\underset{\sim}{x}' = L\underset{\sim}{x} + \underset{\sim}{a}$.]

Exercise: By multiplying two pure boosts along the x-direction, obtain the velocity addition formula.



Let velocity of S_2 w.r.t S_1 be v and $\beta = v/c$,

velocity of S_3 w.r.t S_2 be v' and $\beta' = v'/c$, and

velocity of S_3 w.r.t S_1 be v'' and $\beta'' = v''/c$.

You should get, $\beta'' = \dfrac{\beta + \beta'}{1 + \beta\beta'}$.

A general Lorentz transformation consists of a pure boost and a rotation of the coordinate system.

$$\underbrace{L}_{\text{may not be symmetric}} = \underbrace{L_0}_{\text{pure boost}}\underbrace{R}_{\text{rotation}} = R' L_0'$$