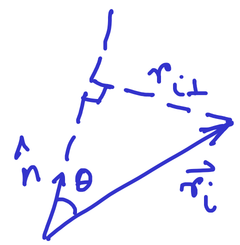
Thus, the kinetic energy is equal to the moment of inertia about the rotation axis times half the magnitude of the angular velocity squared.

$$(\vec{r}_i \times \hat{n})^2 = (\vec{r}_i \times \hat{n})\cdot(\vec{r}_i \times \hat{n}) = \vec{r}_i \cdot \left[\hat{n} \times (\vec{r}_i \times \hat{n})\right] = \vec{r}_i \cdot \left[\vec{r}_i(\hat{n}\cdot\hat{n}) - \hat{n}(\hat{n}\cdot\vec{r}_i)\right]$$

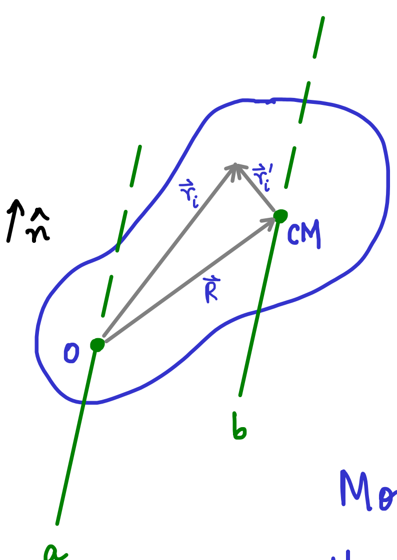
$$= \left[r_i^2 - (\vec{r}_i\cdot\hat{n})^2\right]$$

$\therefore\ I = m_i\,(\vec{r}_i \times \hat{n})^2 = m_i\, r_{i\perp}^2$ ($r_{i\perp}$ = Perpendicular distance of the i^{th} particle from the axis)



$$r_{i\perp} = r_i \sin\theta = |\vec{r}_i \times \hat{n}|$$

The moment of inertia depends on the origin of the body axes.



O = origin of the body axes

CM = Center of mass

$$\vec{r}_i = \vec{r}_i' + \vec{R}$$

Moment of inertia about axis a is

$$I_a = m_i(\vec{r}_i \times \hat{n})^2$$

Moment of inertia about axis b (parallel to a, passing through CM) is

$$I_b = m_i(\vec{r}_i' \times \hat{n})^2$$

$$I_a = m_i(\vec{r}_i \times \hat{n})^2 = m_i\left[(\vec{r}_i' + \vec{R}) \times \hat{n}\right]^2 = m_i\left(\vec{r}_i' \times \hat{n} + \vec{R} \times \hat{n}\right)^2$$

$$= m_i(\vec{r}_i' \times \hat{n})^2 + M(\vec{R} \times \hat{n})^2 + 2m_i\left[(\vec{r}_i' \times \hat{n})\cdot(\vec{R} \times \hat{n})\right], \quad M = \sum_i m_i$$

$$= I_b + MR_{ab}^2 + 2\left[\underbrace{(m_i\vec{r}_i{}')}_{=0}\times\hat{n}\right]\cdot(\vec{R}\times\hat{n})$$

$= 0$ by definition $\because \{\vec{r}_i{}'\}$ are positions w.r.t. the center of mass.

$$\therefore \boxed{I_a = I_b + MR_{ab}^2}$$

$R_{ab} = |\vec{R}\times\hat{n}|$ = distance between the axes a and b.

$\Rightarrow$ Moment of inertia about any axis = moment of inertia about a parallel axis passing through the CM plus the moment of inertia of a particle whose mass is same as that of the body and located at the CM, about the original axis

---

Principal axes

The moment of inertia tensor about any frame is a symmetric matrix. This matrix can be diagonalized. The directions of the eigenvectors are called the principal axes. The diagonalized moment of inertia are called the principal moment of inertia tensor.

$$\overleftrightarrow{I}_D = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} = S^{-1}\overleftrightarrow{I}S \Rightarrow \overleftrightarrow{I} = S\overleftrightarrow{I}_D S^{-1} = S\overleftrightarrow{I}_D S^T$$

$$T = \frac{1}{2}\vec{\omega}^T\overleftrightarrow{I}\vec{\omega} = \frac{1}{2}\vec{\omega}^T\left(S\overleftrightarrow{I}_D S^{-1}\right)\vec{\omega} = \frac{1}{2}\left(S^T\vec{\omega}\right)^T\overleftrightarrow{I}_D\left(S^T\vec{\omega}\right)$$

$\therefore$ If $S^T\vec{\omega} = \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}$, then,

$$T = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2$$

Similarly, $I = (\hat{n})^T\overleftrightarrow{I}\hat{n} = (S^T\hat{n})^T\overleftrightarrow{I}_D(S^T\hat{n}) = I_\alpha n_\alpha^2$

Geometrical Interpretation:

In general,

$$I = n_\alpha I_{\alpha\beta} n_\beta = n_x^2 I_{xx} + n_y^2 I_{yy} + n_z^2 I_{zz} + 2n_y n_z I_{yz} + 2n_z n_x I_{zx} + 2n_x n_y I_{xy}$$

Define $\vec{\rho} = \dfrac{\hat{n}}{\sqrt{I}}$

$$\therefore I_{xx}\rho_x^2 + I_{yy}\rho_y^2 + I_{zz}\rho_z^2 + 2I_{yz}\rho_y\rho_z + 2I_{zx}\rho_z\rho_x + 2I_{xy}\rho_x\rho_y = 1$$

This is the equation of an ellipsoid in the (ρ_x, ρ_y, ρ_z) space. This is the ellipsoid of inertia.

This ellipsoid takes its normal form when the (ρ_x, ρ_y, ρ_z) are transformed using the principal axes transformation S.

i.e., for $\vec{\rho}\,' = S^T\vec{\rho}$, the ellipsoid becomes,

$$I_{xx}\rho_x'^2 + I_{yy}\rho_y'^2 + I_{zz}\rho_z'^2 = 1$$

Radius of gyration

The radius of gyration, R_0, is defined using, $I \equiv MR_0^2 \Rightarrow R_0 = \sqrt{I/M}$

The radius vectors to the points on the ellipsoid of inertia are

$$\vec{\rho} = \frac{\hat{n}}{R_0\sqrt{M}} \Rightarrow |\vec{\rho}| \propto \frac{1}{R_0}$$

i.e. the size of the ellipsoid of inertia is inversely proportional to the radius of gyration.

Solving Rigid Body Problems

Two situations where the problem of rigid body motion may be "easily" solved:

1. There is a point on the rigid body that is fixed in an inertial reference frame — The problem simplifies to rotations about that fixed point.

2. As discussed earlier, the kinetic energy can always be broken up into two parts: $T = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$

$\frac{1}{2}Mv^2$: M- Total mass of the body; v - Velocity of the center of mass.

$\frac{1}{2}I\omega^2$: Rotational kinetic energy about the center of mass.

If the potential energy has a similar break-up, then the problem simplifies.

In both the above situations, we can use the Newtonian approach for the rotational problem.

$$\left(\frac{d\vec{L}}{dt}\right)_s = \vec{\tau} \qquad \text{[Newton's second law for rotations]}$$

$$\Rightarrow \frac{d\vec{L}}{dt} + \vec{\omega}\times\vec{L} = \vec{\tau}$$

where we have denoted the rate of change of angular momentum in the body frame by $\frac{d\vec{L}}{dt}$.

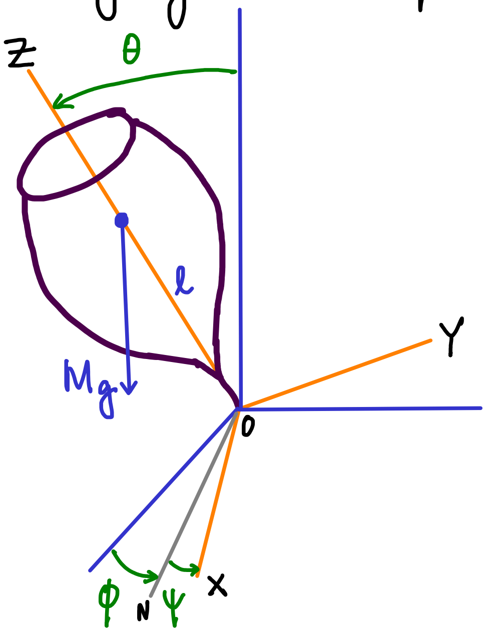
$$\Rightarrow \frac{d}{dt}(\overleftrightarrow{I}\omega) + \vec{\omega}\times\left(\overleftrightarrow{I}\vec{\omega}\right) = \vec{\tau}$$

If the body axes are chosen as the principal axes about the reference point, then, $\overleftrightarrow{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$, and we have,

$$\boxed{\begin{aligned} I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) &= \tau_1 \\ I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) &= \tau_2 \\ I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) &= \tau_3 \end{aligned}}$$

$\leftarrow$ Euler's equations of motion for a rigid body with one point fixed.

Heavy Symmetrical Top with one point fixed



Euler angles : $\theta \to$ Inclination of z-axis from vertical
$\phi \to$ Azimuthal angle about vertical
$\psi \to$ Rotation angle about top's z-axis

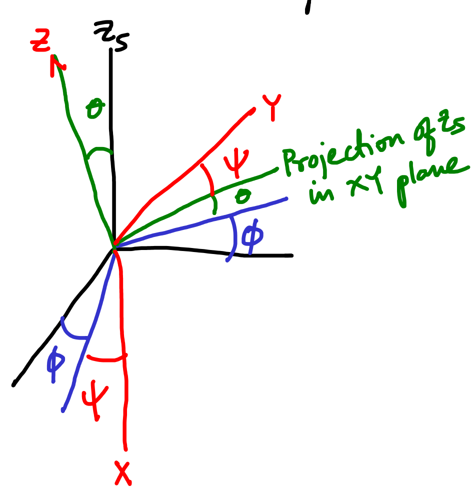
$\dot{\psi}$ = Rotation / spinning about top's z-axis
$\dot{\phi}$ = Precession of top's z-axis about vertical
$\dot{\theta}$ = Nutation of top's z-axis relative to vertical

For a top and a gyroscope, $|\dot{\psi}| \gg |\dot{\theta}| \gg |\dot{\phi}|$ (Spinning $\gg$ Nutation $\gg$ Precession), and $I_1 = I_2 \neq I_3$ (The principal axis can be chosen as the X, Y and Z axes.)

$$\therefore \quad I_1\dot{\omega}_1 - \omega_2\omega_3(I_1 - I_3) = \tau_1$$
$$I_1\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = \tau_2$$
$$I_3\dot{\omega}_3 = \tau_3$$

$\dot{\theta}$ acts along ON. Its components are $(\dot{\theta}\cos\psi, -\dot{\theta}\sin\psi, 0)$

$\dot{\phi}$ acts along the vertical direction. Its z-component is $\dot{\phi}\cos\theta$. Its component on the XY-plane is $\dot{\phi}\sin\theta \Rightarrow$ Its x and y components, respectively, are, $(\dot{\phi}\sin\theta\sin\psi, \dot{\phi}\sin\theta\cos\psi, \dot{\phi}\cos\theta)$



$\dot{\psi}$ acts along z.

$\therefore$ Collecting all the components,

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$

$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

The magnitude of the torque is $Mgl\cos\theta$. It acts in the direction of $\dot{\theta} \Rightarrow \tau_1 = Mgl\cos\theta\cos\psi$, $\tau_2 = -Mgl\cos\theta\sin\psi$, $\tau_3 = 0$

$\because \tau_3 = 0$, $\dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{constant}$.

Let's think of this problem in terms of its Lagrangian.

The kinetic energy,

$$T = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_1\omega_2^2 + \frac{1}{2}I_3\omega_3^2$$

$$= \frac{I_1}{2}\left[\left(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right)^2 + \left(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right)^2\right] + \frac{I_3}{2}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2$$

$$= \frac{I_1}{2}\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{I_3}{2}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2$$

$$V = Mgl\cos\theta$$

$\therefore$ The Lagrangian,

$$\mathcal{L} = T - V = \frac{I_1}{2}\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{I_3}{2}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 - Mgl\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}}\right) = \frac{\partial\mathcal{L}}{\partial\psi} = 0 \Rightarrow \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \text{constant}$$

$$\Rightarrow I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right) = I_3\omega_3 = \text{constant} \equiv I_1 a \quad ---(1)$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) = \frac{\partial\mathcal{L}}{\partial\phi} = 0 \Rightarrow \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \text{constant}$$

$$\Rightarrow I_1\dot{\phi}\sin^2\theta + I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\cos\theta = \text{constant} \equiv I_1 b \quad ---(2)$$

From (1) and (2), $\dot{\phi}$ and $\dot{\psi}$ can be expressed as functions of θ.

$$\text{Eq.}(1)\times\cos\theta - \text{Eq.}(2) \Rightarrow -I_1\dot{\phi}\sin^2\theta = I_1 a\cos\theta - I_1 b$$

$$\Rightarrow \quad \dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} \quad \text{---(3)}$$

Plugging Eq. (3) into Eq. (1), we get,

$$\dot{\psi} = \frac{I_1 a}{I_3} - \left(\frac{b - a\cos\theta}{\sin^2\theta}\right)\cos\theta \quad \text{---(4)}$$

Since the system is conservative, we have another conserved quantity — the energy.

$$E = T + V = \frac{I_1}{2}\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \underbrace{\frac{I_3}{2}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2}_{\frac{1}{2}I_3\omega_3^2} + Mgl\cos\theta$$

$$\Rightarrow \quad E - \frac{1}{2}I_3\omega_3^2 = \text{constant} \equiv \frac{1}{2}I_1\alpha$$

$$\beta \equiv 2\frac{Mgl}{I_1}$$

$$\therefore \quad \frac{1}{2}I_1\alpha = \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_1\left(\frac{b - a\cos\theta}{\sin^2\theta}\right)^2\sin^2\theta + \frac{1}{2}I_1\beta\cos\theta$$

$$\dot{\theta}^2\sin^2\theta = \sin^2\theta\left(\alpha - \beta\cos\theta\right) - \left(b - a\cos\theta\right)^2$$

Let $u = \cos\theta \Rightarrow \dot{u} = -\sin\theta\,\dot{\theta}$

$$\Rightarrow \quad \dot{u}^2 = (1-u^2)(\alpha - \beta u) - (b - au)^2 \quad \text{---(5)}$$

$$\Rightarrow \quad t = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{(1-u^2)(\alpha-\beta u) - (b-au)^2}} \quad \text{---(6)}$$

Thus, in principle, we have $u(t) \Rightarrow \theta(t)$ for all times.
Plugging this into Eq. (3), we have $\phi(t)$ at all times.
Plugging this into Eq. (4), we have $\psi(t)$ at all times.

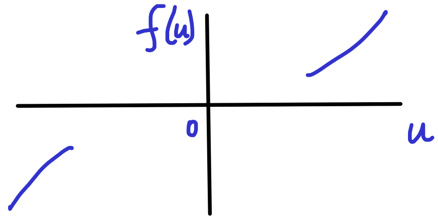
Let's focus on Eq. (5). It is of the form $\dot{u}^2 = f(u)$.

where $f(u) = (1-u^2)(\alpha - \beta u) - (b-au)^2$

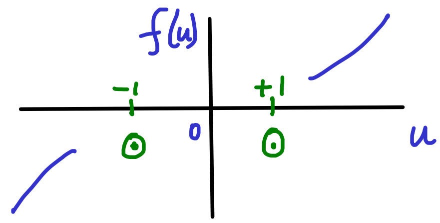
$$= \beta u^3 - (\alpha + a^2)u^2 + (2ab - \beta)u + (\alpha - b^2)$$

The roots of $f(u)$ give us the values of θ where $\dot{\theta}$ changes sign.

For large $|u|$, $f(u) \sim \beta u^3$ [$\beta > 0$ for a top.]

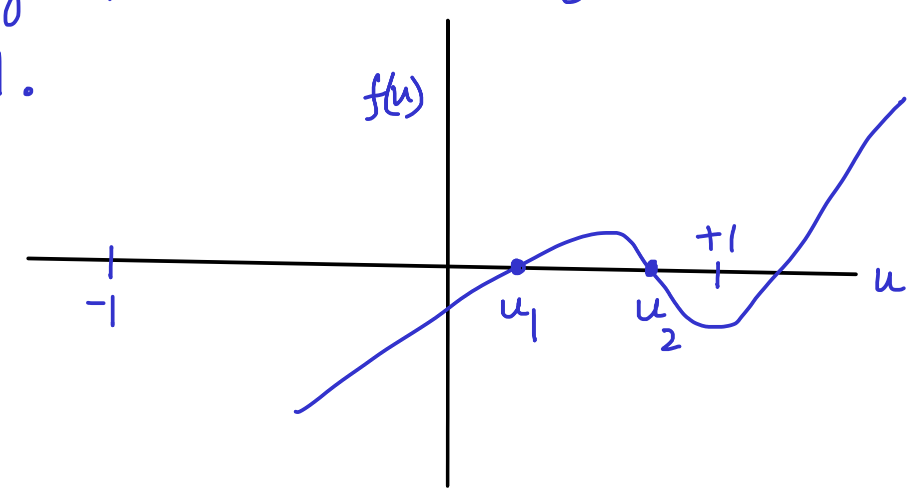


For $u = \pm 1$, $f(u) = -(b \mp a)^2 < 0$



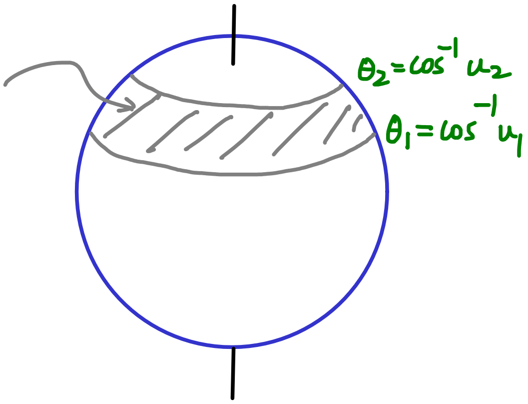
$\Rightarrow$ There must be at least one root for $u > 1$. (unphysical region.)

Since there must be some values of θ corresponding to the physical region, there must always be two roots of $f(u)$ for $-1 < u < 1$.

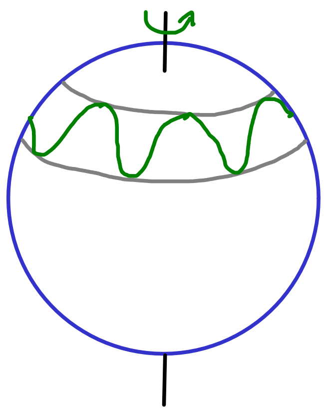


Thus, θ always takes values between $\cos^{-1} u_1$ and $\cos^{-1} u_2$

Allowed values of θ.

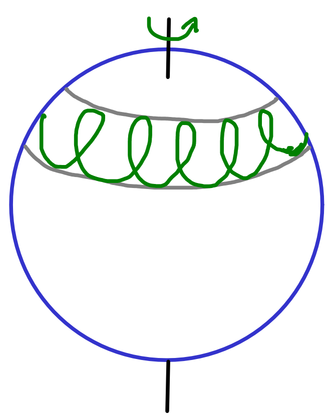


Suppose the initial condition is such that $b > a u_2$, then from Eq.(3), $\dot{\phi}$ is always positive $\Rightarrow$ The top keeps precessing in the same direction. In addition, it nutates with θ in the range between $\cos^{-1} u_1$ and $\cos^{-1} u_2$.

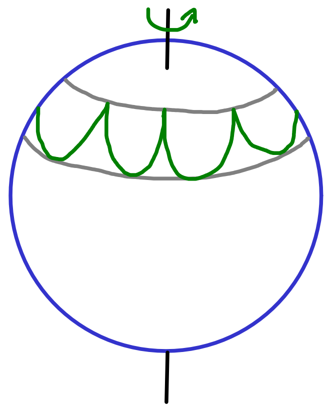


If b/a lies between u_1 and u_2, then,

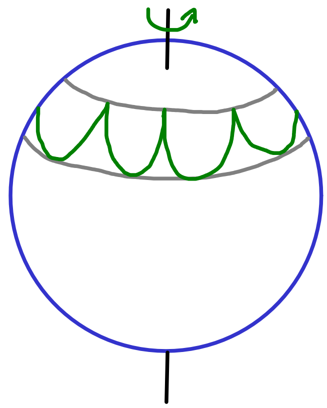
$\dot{\phi} = \dfrac{b - a u}{\sin^2\theta}$ has opposite signs for $\theta = \theta_1$ & $\theta = \theta_2$.



If b/a is a root of $f(u)$ i.e. $b/a = u_1$ or $b/a = u_2$, then $\dot{\phi}$ vanishes on one of the bounding circles $(\theta = \theta_1$ or $\theta = \theta_2)$.



This situation is seen if a spinning top is kept on the ground at an angle θ with zero initial precession. As the top "falls", $\dot{\theta}$ increases and along with it $\dot{\phi}$ also increases.



This situation is seen if a spinning top is kept on the ground at an angle θ with zero initial precession. As the top "falls", $\dot{\theta}$ increases and along with it $\dot{\phi}$ also increases.

If $\frac{1}{2} I_3 \omega_3^2 \gg Mgl$, then we can make some progress analytically.

If u_2 is the initial value of $\cos\theta$, then, since it is equal to b/a, we have, $\alpha = \beta u_2 \quad \because f(u_2) = 0$

$$\therefore f(u) = (1-u^2)(\alpha - \beta u) - (b - au)^2$$
$$= \beta(1-u^2)(u_2 - u) - a^2(u_2 - u)^2$$
$$= (u_2 - u)\left[\beta(1-u^2) - a^2(u_2 - u)\right]$$

$\therefore$ The other root of $f(u)$ must make the quantity in the square brackets vanish.

$$\Rightarrow \beta(1-u_1^2) - a^2(u_2 - u_1) = 0$$

Let $x_1 = u_2 - u_1 \Rightarrow u_1 = u_2 - x_1 \qquad x_1 > 0 \because u_2 > u_1$

$$\Rightarrow \beta\left(1 - u_2^2 - x_1^2 + 2u_2 x_1\right) - a^2 x_1 = 0$$
$$\Rightarrow \beta x_1^2 + (a^2 - 2u_2\beta)x_1 - \beta(1-u_2^2) = 0$$
$$\Rightarrow x_1^2 + \left(\frac{a^2}{\beta} - 2u_2\right)x_1 - (1-u_2^2) = 0$$

For a heavy symmetric top with $\frac{1}{2}I_3\omega_3^2 \gg Mgl$

$$\frac{a^2}{\beta} = \frac{\left(I_3\omega_3/I_1\right)^2}{2Mgl/I_1} = \frac{I_3}{I_1}\frac{\frac{1}{2}I_3\omega_3^2}{Mgl} \gg 1 \qquad \text{if } I_3 \not\ll I_1$$

$$x_1^2 + \frac{a^2}{\beta}x_1 - (1-u_2^2) \approx 0$$

$$\Rightarrow \quad x_1 \approx \frac{-a^2/\beta \pm \sqrt{a^4/\beta^2 + 4(1-u_2^2)}}{2}$$

$$\approx \frac{-a^2/\beta \pm \frac{a^2}{\beta}\left[1 + \frac{2\beta^2}{a^4}(1-u_2^2)\right]}{2}$$

$$\approx \frac{\beta}{a^2}(1-u_2^2) \qquad \because \quad x_1 > 0$$

$$\approx 0$$

$\therefore \Rightarrow u_1 \approx u_2 \Rightarrow$ The top has negligible nutation.

$$f(u) = (u_2-u)\left[\beta(1-u^2) - a^2(u_2-u)\right]$$

$$= x\left[\beta\left(1-(u_2-x)^2\right) - a^2x\right] \qquad x \equiv u_2 - u$$

$$= a^2x\left[\frac{\beta}{a^2}(1-u_2^2) - x - \underbrace{\frac{\beta}{a^2}(x^2-2u_2x)}_{\ll x}\right] \quad \because \beta/a^2 \ll 1$$

$$\approx a^2x(x_1-x)$$

Let $y = x - x_1/2$, i.e. as x goes from 0 to x_1, y goes from $-\frac{x_1}{2}$ to $\frac{x_1}{2}$.

$$\dot{y}^2 = \dot{x}^2 = \dot{u}^2 = f(u) = a^2(y+x_1/2)(x_1-y-x_1/2) = -a^2(y+x_1/2)(y-x_1/2)$$

$$\Rightarrow \dot{y}^2 = -a^2y^2 + a^2x_1^2/4$$

$\Rightarrow \quad 2\dot{y}\ddot{y} = -2a^2 y\dot{y}$

$\Rightarrow \quad \ddot{y} = -a^2 y \quad \Rightarrow \quad y = A\cos at + B\sin at$

$\because \; x=0$ and $\dot{x}=0$ at $t=0 \Rightarrow y = -x_1/2$ and $\dot{y}=0$ at $t=0$.

$\Rightarrow \quad A = -x_1/2 \quad$ and $B=0$

$\Rightarrow \quad x = \frac{x_1}{2}(1-\cos at)$

$\Rightarrow$ a is the angular frequency of nutation (It is proportional to ω_3 — spin frequency $\because \; I_1 a = I_3\omega_3$.)

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} = \frac{a(u_2 - u)}{\sin^2\theta} \approx \frac{ax}{\sin^2\theta_2} = \frac{ax}{1-u_2^2} = \frac{ax}{a^2 x_1/\beta}$$

$$\Rightarrow \dot{\phi} = \frac{\beta}{a}\frac{x}{x_1} = \frac{\beta}{2a}(1-\cos at)$$

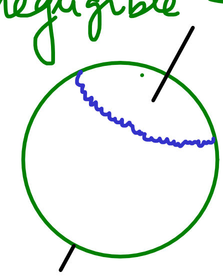
$\therefore$ The precession angular velocity oscillates around $\beta/2a$.

$$\frac{\beta}{2a} = \frac{2Mgl/I_1}{2I_3\omega_3/I_1} = \frac{Mgl}{I_3\omega_3} = \frac{Mgl}{I_3\omega_3^2}\,\omega_3 \ll \omega_3 .$$

$\therefore$ The top precesses slowly.

Summary:

A fast top released at an angle with zero intial precession, starts to fall, picks up slow precession and fast nutation, but the extent of nutation is negligible — Pseudo regular motion.



Regular motion → No nutation.

$\theta_1 = \theta_2 \Rightarrow$ The initial value of θ corresponds to a double root of $f(u)$. This can happen for some choices of the initial conditions.

Vertical spinning:

$u=1$ must be a root. This happens when $a=b$.

$$E - \frac{1}{2} I_3 \omega_3^2 = \frac{1}{2} I_1 \alpha$$

$$\Rightarrow \frac{1}{2} I_1 \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + Mgl\cos\theta = \frac{1}{2} I_1 \alpha$$

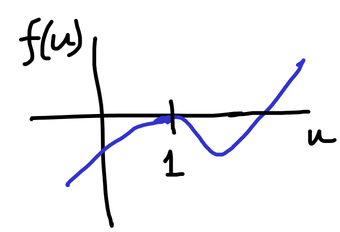
Initially, $\theta = 0$, $\dot{\theta} = 0$. $\Rightarrow Mgl = \frac{1}{2} I_1 \alpha \Rightarrow \alpha = \beta$

$$\begin{aligned} f(u) &= (1-u^2)(\alpha - \beta u) - (b - au)^2 \\ &= \beta (1-u^2)(1-u) - a^2 (1-u)^2 \\ &= (1-u)^2 \left[\beta(1+u) - a^2\right] \end{aligned}$$

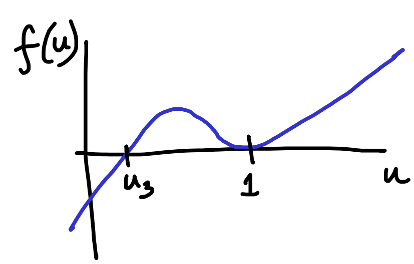
∴ The third root satisfies

$$\beta(1+u_3) - a^2 = 0 \Rightarrow u_3 = a^2/\beta - 1$$

If $u_3 \geqslant 1$, we have pure vertical spin.



If $u_3 < 1$, the top nutates with $u_3 \leqslant u \leqslant 1$.



∴ There is a critical frequency required for pure vertical spins.

$u_3 = 1$ corresponds to this frequency

$$\Rightarrow \frac{a^2}{\beta} - 1 = 1$$

$$\Rightarrow a^2 = 2\beta$$

$$\Rightarrow \frac{I_3^2 \omega_*^2}{I_1^2} = \frac{4Mgl}{I_1} \Rightarrow \omega_*^2 = \frac{4MglI_1}{I_3^2}$$

For $\omega_3 > \omega_*$ we can have pure vertical spinning.

In a real top if this is satisfied initially, as the top slows down, this is no longer satisfied and the top starts to wobble.