The eigenvalues of the rotation matrix A: $\{1, e^{\pm i\Phi}\}$

Φ represents the rotation angle.

Proof: Let's move to a basis where the rotation axis is along the z-direction, i.e. we rotate our reference frame with matrix S, so that in the rotated frame, the z-direction is the axis of rotation.

X = vector in original frame

A = rotation matrix being considered

$X' = AX$ = rotated vector using A.

$$\Rightarrow S^{-1}X' = S^{-1}AX$$
$$= S^{-1}ASS^{-1}X$$
$$\Rightarrow X'_S = A_S X_S$$

where $A_S \equiv S^{-1}AS$ = A in frame rotated using S

$X_S \equiv S^{-1}X$ = X in frame rotated using S

$X'_S \equiv S^{-1}X'$ = X' in frame rotated using S

Since S is chosen so that A becomes a rotation along the z-axis,

$$A_S = \begin{pmatrix} \cos\Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \mathrm{Tr}\, A_S = 2\cos\Phi + 1$$

$$\mathrm{Tr}\, A_S = \mathrm{Tr}\, S^{-1}AS = \mathrm{Tr}(SS^{-1}A) \quad \text{(cyclic property of trace)}$$
$$= \mathrm{Tr}\, A$$
$$\Rightarrow \mathrm{Tr}\, A = 2\cos\Phi + 1$$

$$\Rightarrow 1 + e^{i\Phi} + e^{-i\Phi} = 2\cos\Psi + 1$$

$$\Rightarrow \quad \cos\Phi = \cos\Psi$$

$$\Rightarrow \quad \Phi = \pm\Psi$$

$\because$ Rotation by $-\Psi$ corresponds to rotation by $+\Psi$ about an axis in the opposite direction, and since the sign of the eigenvector is arbitrary, without loss of generality, $\Phi = \Psi$.

Charles' theorem: The most general displacement of a rigid body is a translation plus a rotation.

Thus, the six independent coordinates of a rigid body $(x, y, z, \psi, \theta, \phi)$.

Relating Φ to the Euler angles (ψ, θ, ϕ):

$$\mathrm{Tr}\, A = 1 + 2\cos\Phi = \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi - \sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi + \cos\theta$$

$$\Rightarrow 4\cos^2\Phi/2 - 1 = \cos(\psi+\phi) + \cos\theta\cos(\psi+\phi) + \cos\theta$$

$$\Rightarrow 4\cos^2\Phi/2 - 1 = 2\cos^2\left(\frac{\psi+\phi}{2}\right) - 1 + \left(2\cos^2\frac{\theta}{2} - 1\right)\left(2\cos^2\left(\frac{\psi+\phi}{2}\right) - 1\right) + 2\cos^2\frac{\theta}{2} - 1$$

$$\Rightarrow 4\cos^2\frac{\Phi}{2} = 4\cos^2\frac{\theta}{2}\cos^2\left(\frac{\psi+\phi}{2}\right)$$

$$\Rightarrow \cos\frac{\Phi}{2} = \pm\cos\frac{\theta}{2}\cos\left(\frac{\psi+\phi}{2}\right)$$

As $(\psi, \theta, \phi) \to (0,0,0)$, Φ must tend towards $0 \Rightarrow \cos\frac{\Phi}{2}$ must tend towards $+1$.

$$\therefore \quad \cos\frac{\Phi}{2} = \cos\frac{\theta}{2}\cos\frac{\psi+\phi}{2}$$

We had seen earlier that an infinitesimal rotation can be represented by

$$A = \mathbb{1} + \epsilon \;, \quad \text{where } \epsilon_{ij} \ll 1$$

$$\Rightarrow A^{-1} = \mathbb{1} - \epsilon \qquad \because (\mathbb{1}+\epsilon)(\mathbb{1}-\epsilon) = \mathbb{1} + O(\epsilon^2)$$

We know that A is orthogonal.

$$\therefore (\mathbb{1}+\epsilon)^T = \mathbb{1} - \epsilon \Rightarrow \epsilon^T = -\epsilon \quad \text{(skew symmetric)}$$

Let

$$\epsilon = \begin{pmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{pmatrix}$$

Since $\vec{r}\,' = A\vec{r}$,

$$d\vec{r} = \vec{r}\,' - \vec{r} = \epsilon\vec{r} = \begin{pmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \begin{cases} dx_1 = x_2\, d\Omega_3 - x_3\, d\Omega_2 \\ dx_2 = x_3\, d\Omega_1 - x_1\, d\Omega_3 \\ dx_3 = x_1\, d\Omega_2 - x_2\, d\Omega_1 \end{cases}$$

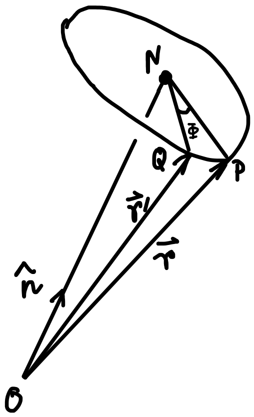
If $d\vec{\Omega}$ is a vector with components $(d\Omega_1, d\Omega_2, d\Omega_3)$, then we can write,

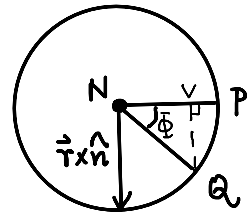
$$d\vec{r} = \vec{r} \times d\vec{\Omega} \,.$$

Aside: Let's see how $\vec{r}$ changes due to a rotation around $\hat{n}$ with angle Φ.

Consider rotation of a coordinate system about the direction $\hat{n}$ with an angle Φ. This is equivalent to rotation by $-\Phi$, for any position vector $\vec{r}$ around the same direction.

$\rightarrow$ passive rotation by $\Phi \Leftrightarrow$ active rotation by $-\Phi$.





$$\overrightarrow{ON} = (\vec{r}.\hat{n})\hat{n}$$

$$\overrightarrow{NP} = \overrightarrow{OP} - \overrightarrow{ON} = \vec{r} - (\vec{r}.\hat{n})\hat{n}$$

$$|\vec{r}\times\overrightarrow{ON}| = 2\times \text{area of } \triangle ONP$$

$$\Rightarrow (\vec{r}.\hat{n})\,|\vec{r}\times\vec{n}| = 2\times\frac{1}{2}|\overrightarrow{ON}||\overrightarrow{NP}|$$

$$= \vec{r}.\hat{n}\,|\overrightarrow{NP}| \Rightarrow |\vec{r}\times\vec{n}| = |\overrightarrow{NP}|$$

Also, $|\overrightarrow{NQ}| = |\overrightarrow{NP}|$ (from figure)

$$\vec{r}' = \overrightarrow{OQ} = \overrightarrow{ON} + \overrightarrow{NV} + \overrightarrow{VQ} = (\vec{r}.\hat{n})\hat{n} + \overrightarrow{NP}\cos\Phi + (\vec{r}\times\hat{n})\sin\Phi$$

$$\Rightarrow \vec{r}' = (\vec{r}.\hat{n})\hat{n} + [\vec{r} - (\vec{r}.\hat{n})\hat{n}]\cos\Phi + (\vec{r}\times\hat{n})\sin\Phi$$

$$\Rightarrow \boxed{\vec{r}' = \vec{r}\cos\Phi + (\vec{r}.\hat{n})\hat{n}(1-\cos\Phi) + (\vec{r}\times\hat{n})\sin\Phi}$$

$\Rightarrow$ For rotation by a small angle $d\Phi$,

$$\vec{r}' = \vec{r} + (\vec{r}\times\hat{n})d\Phi \Rightarrow d\vec{r} = (\vec{r}\times\hat{n})\,d\Phi$$

$$\therefore \quad d\vec{\Omega} = \hat{n}\,d\Phi$$

Rate of change of a vector

Let $\vec{G}$ be some vector.

Consider a reference frame S fixed in space and another frame S' which is attached to a rigid body which has one fixed point, chosen as the origin of S'.

$$\left(d\vec{G}\right)_{space} = \text{Change in } \vec{G} \text{ measured in } S$$

$$\left(d\vec{G}\right)_{body} = \text{Change in } \vec{G} \text{ measured in } S'$$

If S' has axes parallel to S, then,

$\left(d\vec{G}\right)_{space} = \left(d\vec{G}\right)_{body}$. [$\because$ $\vec{G}$ and $\left(\vec{G}+d\vec{G}\right)$ differ by the same amounts]

If not, $\left(d\vec{G}\right)_{space} \neq \left(d\vec{G}\right)_{body}$ and we write,

$$\left(d\vec{G}\right)_{space} = \left(d\vec{G}\right)_{body} + \left(d\vec{G}\right)_{rot}$$

If $\vec{G}$ is fixed in the body frame, $\left(d\vec{G}\right)_{body} = 0$. In this case, $\left(d\vec{G}\right)_{space}$ is due to rotation of the body and therefore active rotation of the vector $\vec{G}$ in the anticlockwise direction. For a passive anticlockwise rotation, $d\vec{G} = \vec{G} \times d\vec{\Omega}$.

$\therefore$ For an active anticlockwise rotation, we have,

$$\left(d\vec{G}\right)_{space} = d\vec{\Omega} \times \vec{G} = \left(d\vec{G}\right)_{rot}$$

$\therefore$ In general, $\left(d\vec{G}\right)_{space} = \left(d\vec{G}\right)_{body} + d\vec{\Omega} \times \vec{G}$

$$\Rightarrow \boxed{\left(\frac{d\vec{G}}{dt}\right)_{space} = \left(\frac{d\vec{G}}{dt}\right)_{body} + \vec{\omega}\times\vec{G}}, \text{ where } \vec{\omega} = \frac{d\vec{\Omega}}{dt}$$

$\vec{\omega}$ should not be thought of as the derivative of $\vec{\Omega}$, but $\vec{\omega}\,dt$ should be interpreted as a vector whose magnitude is the angle of rotation in time dt and whose direction is the instantaneous axis of rotation.

Angular Momentum and kinetic energy of motion about a point

A rigid body is described by six coordinates $(x, y, z, \psi, \theta, \phi)$, where (x, y, z) are the coordinates of some point in the body and (ψ, θ, ϕ) describe the orientation of the body w.r.t. a frame whose origin is the point (x, y, z).

If this origin of the body reference frame is chosen to be the center of mass of the rigid body, then, we can write the kinetic energy of the rigid body as,

$$T = \frac{1}{2}Mv^2 + T'(\psi, \theta, \phi)$$

where M = total mass, v = velocity of the center of mass

T' = kinetic energy due to orientational degrees of freedom.

In problems where the potential energy also separates into terms involving translational and orientational degrees of freedom, we can solve the translational and orientational problems separately.

Examples: ① Rigid body in uniform gravitational field: $V = -mgz$ [independent of (ψ, θ, ϕ)]

② Rigid body in uniform magnetic field: $V \propto -\vec{M}\cdot\vec{B} = -MB\cos\theta$ [independent of (x, y, z)]

We will therefore try to understand the problem involving rigid body motion, with one point fixed.

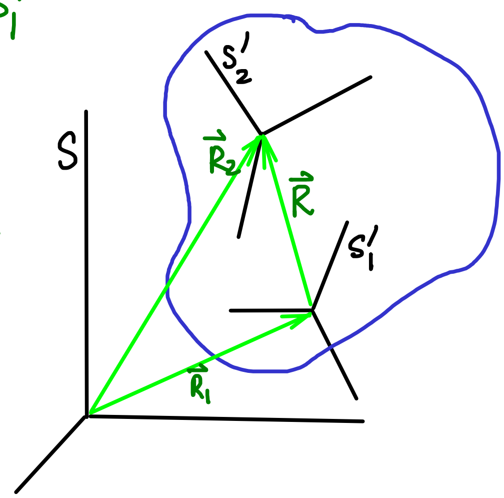
The rotation angle and the angular velocity are independent of the origin of the body frame

Proof: Consider a fixed external reference frame S and two body reference frames S_1' and S_2', whose origins are at $\vec{R}_1$ and $\vec{R}_2$ respectively in frame S.

The vector $\vec{R} = \vec{R}_2 - \vec{R}_1$ is a fixed vector in S_1'

$$\therefore \left(\frac{d\vec{R}}{dt}\right)_S = \vec{\omega}_1 \times \vec{R}$$

where $\vec{\omega}_1$ is the angular velocity vector for the body frame S_1'.



Using the same logic for the body frame S_2' and the vector $-\vec{R} = \vec{R}_1 - \vec{R}_2$, we have,

$$\left(\frac{d(-\vec{R})}{dt}\right)_S = \vec{\omega}_2 \times (-\vec{R})$$

where $\vec{\omega}_2$ is the angular velocity vector for the body frame S_2'

$$\therefore \vec{\omega}_1 \times \vec{R} = \vec{\omega}_2 \times \vec{R} \Rightarrow (\vec{\omega}_1 - \vec{\omega}_2) \times \vec{R} = 0$$

$\Rightarrow$ $\vec{\omega}_1$ and $\vec{\omega}_2$ can have nonzero difference only along the line joining the two chosen origins.

Since this choice was arbitrary, the above result must be true for every pair of points in the rigid body.

This can only be true if $\vec{\omega}_1 = \vec{\omega}_2 = \vec{\omega}$.

Suppose a rigid body moves with one stationary point.
Then, the angular momentum about that point,

$$\vec{L} = m_i (\vec{r}_i \times \vec{v}_i) \quad , \text{ where } \begin{cases} \text{① summation convention has been assumed.} \\ \text{② } \vec{r}_i \text{ is the position of the } i^{th} \text{ particle relative to the stationary point.} \\ \text{③ } \vec{v}_i = \left(\frac{d\vec{r}_i}{dt}\right)_s \end{cases}$$

Since the position vectors of the points of the rigid body are fixed in the body frame, $\vec{v}_i = \vec{\omega} \times \vec{r}_i$

$$\Rightarrow \vec{L} = m_i \left[\vec{r}_i \times (\vec{\omega} \times \vec{r}_i) \right] = m_i r_i^2 \vec{\omega} - m_i \vec{r}_i (\vec{\omega} \cdot \vec{r}_i)$$

$$\Rightarrow L_x = m_i (r_i^2 - x_i^2) \omega_x - m_i x_i y_i \omega_y - m_i x_i z_i \omega_z$$

$$= I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z$$

Similarly, $L_y = I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z$
and $L_z = I_{zx} \omega_x + I_{zy} \omega_y + I_{zz} \omega_z$

$I_{xx} = m_i (r_i^2 - x_i^2)$ and the other diagonal terms, I_{yy} and I_{zz} are called the moments of inertia about the x, y and z axes.

$I_{xy} = - m_i x_i y_i$ and the other cross terms are known as the products of inertia.

$$\overleftrightarrow{I} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$ is called the moment of inertia tensor.

$$\vec{L} = \overleftrightarrow{I} \vec{\omega}$$

For continuous systems,

$$I_{xx} = \int d^3r\, \rho(\vec{r})\,(r^2 - x^2)\,, \quad I_{xy} = -\int d^3r\, \rho(\vec{r})\, xy$$

Compact notations:

Discrete:

$$I_{\alpha\beta} = m_i\left(r_i^2 \delta_{\alpha\beta} - x_{i\alpha} x_{i\beta}\right)$$

Continuous:

$$I_{\alpha\beta} = \int d^3r\, \rho(\vec{r})\left(r^2 \delta_{\alpha\beta} - x_\alpha x_\beta\right)$$

where the subscripts α and β take values $(1,2,3)$ which represent the 3 cartesian directions (x,y,z).

Kinetic Energy:

$$\begin{aligned}
T &= \frac{1}{2} m_i v_i^2 \\
&= \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i \\
&= \frac{1}{2} m_i \vec{v}_i \cdot (\vec{\omega} \times \vec{r}_i) \\
&= \frac{1}{2} m_i \vec{\omega} \cdot (\vec{r}_i \times \vec{v}_i) \\
&= \frac{1}{2} \vec{\omega} \cdot \vec{L} \qquad \left[\because \vec{L} = m_i (\vec{r}_i \times \vec{v}_i)\right] \\
&= \frac{1}{2} (\vec{\omega})^T \overleftrightarrow{I}\, \vec{\omega} \qquad \left(\text{also written as } \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega} = \frac{1}{2}\omega_\alpha I_{\alpha\beta}\omega_\beta\right)
\end{aligned}$$

$\vec{\omega} \equiv \omega \hat{n}$, i.e., $\hat{n}$ is the direction of $\vec{\omega}$ and ω is its magnitude.

$$\therefore \quad T = \frac{1}{2}\left((\hat{n})^T \overleftrightarrow{I}\, \hat{n}\right)\omega^2 \equiv \frac{1}{2} I \omega^2$$

where $I \equiv (\hat{n})^T \overleftrightarrow{I}\, \hat{n}$ is the moment of inertia about the rotation axis.

$$\begin{aligned}
I = n_\alpha I_{\alpha\beta} n_\beta &= n_\alpha m_i \left(r_i^2 \delta_{\alpha\beta} - x_{i\alpha} x_{i\beta}\right) n_\beta = m_i\left[r_i^2 n_\alpha n_\alpha - (x_{i\alpha} n_\alpha)^2\right] \\
&= m_i\left[r_i^2 - (\vec{r}_i \cdot \hat{n})^2\right]
\end{aligned}$$