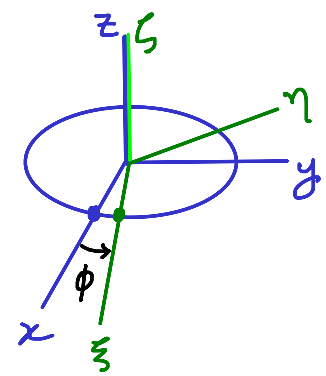
EULER ANGLES

As argued earlier, three angles must be sufficient to describe every rotation. One such set of angles are called Euler angles.
Every rotation from (x,y,z) to (x',y',z') can be broken up into the following elementary rotations.

① Rotate around z-axis by ϕ anticlockwise.

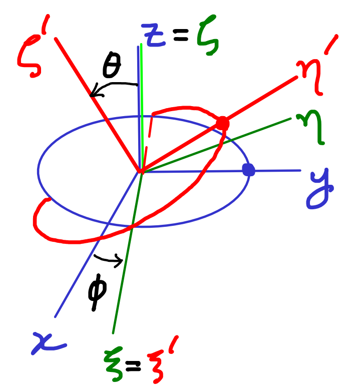
(convention: all positive rotations are taken as anticlockwise in these notes, when not mentioned explicitely.)

The new axes are named $\xi\eta\zeta$.



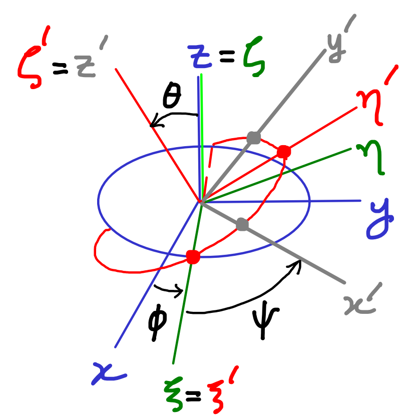
② Rotate around ξ-axis (new x-axis) by θ.

The new axes are named $\xi'\eta'\zeta'$.



③ Rotate around ζ'-axis (new z-axis) by ψ.

The new axes are named $x'y'z'$.



Thus, the order of the rotation axes are ZXZ.

∴ The overall rotation matrix is given by,

$$A = R_z(\psi)\, R_x(\theta)\, R_z(\phi)$$

$$\Rightarrow A = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\cos\theta\sin\phi & \cos\theta\cos\phi & \sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi & \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

Check whether the above matrix is orthogonal.

Possible conventions for the definitions of the Euler angles — The first axis of rotation can be arbitrary (3 choices). The second axis must be different from the first (2 choices). The third must be different from the second (2 choices). Thus, there can be TWELVE different conventions for the definitions of the Euler angles.

In quantum mechanics, ZYZ order of rotations is often used.

In mechanical engineering, XYZ order is used.

<u>Euler's theorem</u>: Any displacement of a rigid body with one point fixed is a rotation about some axis.

In a rotation about some axis, the points on the axis of rotation remain unchanged.

$\Rightarrow \quad AR = R \quad$ if R denotes a point along the axis of rotation.

The above relation implies that R is an eigenvector of A with eigenvalue $+1$.

Conversely, any real orthogonal matrix with eigenvalue $+1$ is a rotation with the eigenvector representing the direction of the rotation axis.

Proof of Euler's theorem from orthogonality:

$$AA^T = \mathbb{1} \Rightarrow (\mathbb{1} - A)A^T = A^T - \mathbb{1}$$

$$\Rightarrow \det(\mathbb{1} - A)\det A = \det(A^T - \mathbb{1})$$

$$\Rightarrow \det(\mathbb{1} - A)\det A = \det(A - \mathbb{1})$$

$$\Rightarrow (-1)^3 \det(A - \mathbb{1})\det A = \det(A - \mathbb{1})$$

[$\because \det(-X) = (-1)^n \det X$ if X is an $(n\times n)$ matrix]

$$\Rightarrow \det(A - \mathbb{1})(\det A + 1) = 0$$

$\because \det A = 1$, $\det(A - \mathbb{1}) = 0$

$\Rightarrow$ A has an eigenvalue of $+1$

$\Rightarrow$ A is a rotation about some axis.

Other eigenvalues:

Since A has real elements, the characteristic equation is a cubic equation with real coefficients. $\Rightarrow$ If λ is a complex eigenvalue of A, λ^* is also an eigenvalue.

Let $AR = \lambda R$

$\Rightarrow R^\dagger A^\dagger = \lambda^* R^\dagger \Rightarrow R^\dagger A^T = \lambda^* R^\dagger$ ($\because$ A is has real elements)

$\Rightarrow R^\dagger A^T A R = \lambda^* R^\dagger A R \Rightarrow R^\dagger R = \lambda^* \lambda R^\dagger R$

$\Rightarrow \lambda^* \lambda = 1$ $\because$ $R^\dagger R$ is the norm of the eigenvector and is not zero.

$\therefore$ The eigenvalues of A are $\{1, e^{\pm i\Phi}\}$

$\Rightarrow$ (1) If $\Phi = 0$, $A = \mathbb{1}$ (rotation by zero angle.)

(2) Other values of Φ correspond to non-trivial rotations.

Φ represents the rotation angle.