RIGID BODIES

SYLLABUS:
Independent coordinates, orthogonal transformations and rotations (finite and infinitesimal), Euler's theorem, Euler angles, Inertia tensor and principal axis system;
Euler's equations, Heavy symmetrical top with precession and notation.

Textbook: Goldstein, Poole, Safko — Classical Mechanics

A rigid body is a system of particles which are allowed to move such that the distance between every pair of particles in the system remains fixed.

Independent coordinates / Degrees of freedom

System with N- particles without any constraints → 3N degrees of freedom.

m - independent constraints ⇒ (3N − m) degrees of freedom.

Rigid bodies:

$N = 2 \rightarrow$ 1 constraint : $|\vec{r}_1 - \vec{r}_2| = c_{12} = \text{constant}$

∴ 5 degrees of freedom.

$N = 3 \rightarrow$ 3 independent constraints → $9 - 3 = 6$ degrees of freedom

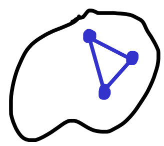
$N = 4 \rightarrow$ First three particles → 3 independent constraints.
Coordinates of the fourth point w.r.t. any of the first three points → 3 independent constraints.

⇒ $12 - 6 = 6$ degrees of freedom

Each additional point brings in no additional degrees of freedom, since when $N \rightarrow N+1$, $3N \rightarrow 3N+3$ and fixing the last point requires 3 new constraints ⇒ Number of independent coordinates dont change.

A rigid body is completely specified by SIX independent coordinates.

Explanation 1: A rigid body is fixed if we fix three non-collinear points in space.

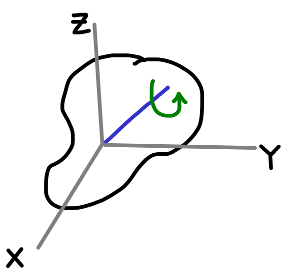


Thus, we need nine coordinates to specify the three points. We also have three constraints that fix the distances between these three points.

Thus, we have $9-3=6$ independent coordinates.

Explanation 2: ① Fix one point of the rigid body (3 coordinates)

The rigid body can move around that point.



② Choose a line passing through the above point. This line can have arbitrary orientation which can be fixed by two coordinates — (θ, ϕ)

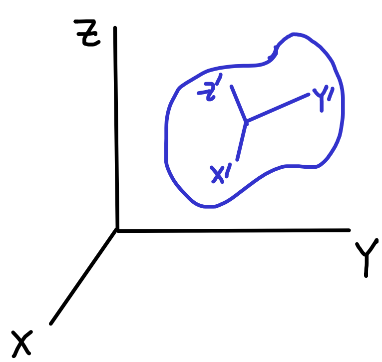
The body can now rotate around this line.

③ The body is completely fixed by one more coordinate that specifies the orientation around the above line.

Specifying the six independent coordinates

Let there a reference frame, S, fixed in space — (X, Y, Z)

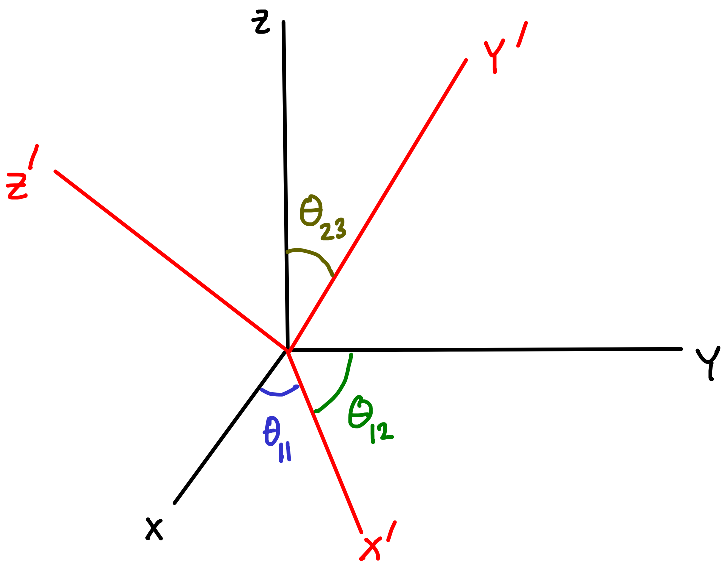
Let there be another reference frame, S', fixed to the rigid body of interest.



The relation between the coordinates (x, y, z) and (x', y', z') can be worked out in two steps — (1) Consider two frames which have the same origin and look at the relation between the coordinates — one of them being S' and the other frame having axes parallel to S.

(2) The relation between the coordinates in S & S' is obtained using a simple shift of all the coordinates obtained in the above step.

Since the second step is trivial, we will discuss only the first step, i.e. relation between coordinates of S and S' when their origins coincide. Moreover, the second step requires three degrees of freedom $\Rightarrow$ The first step must also require three independent degrees of freedom / angles.



There are angles between the axes of S and S'.

θ_{lm} = angle between the l^{th} axis of S' and the m^{th} axis of S.

$$\Rightarrow \cos\theta_{11} = \hat{i}'\cdot\hat{i}, \; \cos\theta_{12} = \hat{i}'\cdot\hat{j}, \; \cos\theta_{13} = \hat{i}'\cdot\hat{k}, \; \cos\theta_{21} = \hat{j}'\cdot\hat{i}, \ldots$$

$$\therefore \hat{i}' = \cos\theta_{11}\,\hat{i} + \cos\theta_{12}\,\hat{j} + \cos\theta_{13}\,\hat{k}$$
$$\hat{j}' = \cos\theta_{21}\,\hat{i} + \cos\theta_{22}\,\hat{j} + \cos\theta_{23}\,\hat{k}$$
$$\hat{i}' = \cos\theta_{31}\,\hat{i} + \cos\theta_{32}\,\hat{j} + \cos\theta_{33}\,\hat{k}$$

(9 angles)

[Similarly, $\hat{i} = \cos\theta_{11}\,\hat{i}' + \cos\theta_{21}\,\hat{j}' + \cos\theta_{31}\,\hat{k}'$

$\hat{j} = \cos\theta_{12}\,\hat{i}' + \cos\theta_{22}\,\hat{j}' + \cos\theta_{32}\,\hat{k}'$

$\hat{i} = \cos\theta_{13}\,\hat{i}' + \cos\theta_{23}\,\hat{j}' + \cos\theta_{33}\,\hat{k}'$]

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = x'\hat{i}' + y'\hat{j}' + z'\hat{k}'$$

$$\Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta_{11} & \cos\theta_{12} & \cos\theta_{13} \\ \cos\theta_{21} & \cos\theta_{22} & \cos\theta_{23} \\ \cos\theta_{31} & \cos\theta_{32} & \cos\theta_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

[$\because x' = \vec{r}\cdot\hat{i}'$,
$= x\cos\theta_{11} + y\cos\theta_{12} + z\cos\theta_{13}$
and so on]

Constraints on $\{\theta_{lm}\}$:

$$\left.\begin{array}{l} \hat{i}'\cdot\hat{i}' = \hat{j}'\cdot\hat{j}' = \hat{k}'\cdot\hat{k}' = 1 \\ \hat{j}'\cdot\hat{k}' = \hat{k}'\cdot\hat{i}' = \hat{i}'\cdot\hat{j}' = 0 \end{array}\right\} \text{ 6 constraints.}$$

$\therefore$ The number of independent degrees of freedom of the rigid body works out correctly $\to 6 = (3$ for origin of $S') + (9-6$ for orientation$)$

Change of notation: $(x, y, z) \Rightarrow (x_1, x_2, x_3)$, $(x', y', z') \Rightarrow (x_1', x_2', x_3')$
$(\hat{i}, \hat{j}, \hat{k}) \Rightarrow (\hat{x}_1, \hat{x}_2, \hat{x}_3)$, $(\hat{i}', \hat{j}', \hat{k}') \Rightarrow (\hat{x}_1', \hat{x}_2', \hat{x}_3')$

$$\cos\theta_{lm} = \hat{x}_l'\cdot\hat{x}_m$$

$$\hat{x}_l' = (\cos\theta_{lm})\,\hat{x}_m \quad \text{(Einstein's convention: sum over repeated indices)}$$

$$\hat{x}_l'\cdot\hat{x}_m' = \delta_{lm}$$

$$\Rightarrow \left((\cos\theta_{lp})\,\hat{x}_p\right)\cdot\left((\cos\theta_{mq})\,\hat{x}_q\right) = \delta_{lm}$$

$$\Rightarrow \left(\cos\theta_{lp}\cos\theta_{mq}\right)\left(\hat{x}_p\cdot\hat{x}_q\right) = \delta_{lm}$$

$$\Rightarrow \left(\cos\theta_{lp}\cos\theta_{mq}\right)\delta_{pq} = \delta_{lm}$$

$$\Rightarrow \left(\cos\theta_{lp}\right)\left(\cos\theta_{mp}\right) = \delta_{lm} \quad ---(1)$$

Using the reverse equations, $\left(\cos\theta_{pl}\right)\left(\cos\theta_{pm}\right) = \delta_{lm}$ ---(2)

Similarly, for the coordinates, we have, $x'_l = \cos\theta_{lm}\, x_m$.

Length-squared of the vector $\vec{x}$ in $S' = x'_l x'_l$

$$= \left(\cos\theta_{lm} x_m\right)\left(\cos\theta_{lp} x_p\right)$$

$$= \cos\theta_{lm}\cos\theta_{lp}\, x_m x_p$$

$$= \delta_{mp}\, x_m x_p$$

$$= x_m x_m$$

= Length-squared in S.

Eq. 1 is known as the orthogonality condition. If we denote the matrix $\begin{pmatrix} \cos\theta_{11} & \cos\theta_{12} & \cos\theta_{13} \\ \cos\theta_{21} & \cos\theta_{22} & \cos\theta_{23} \\ \cos\theta_{31} & \cos\theta_{32} & \cos\theta_{33} \end{pmatrix}$ by A, Eq. 1 implies,

$$A A^T = \mathbb{1}. \qquad (\mathbb{1} \text{ is the identity matrix.})$$

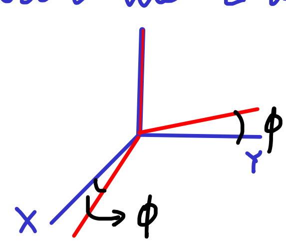
Similarly, Eq. 2 implies $A^T A = \mathbb{1}$.

The matrix A is an example of a rotation matrix.

E.g. Consider two-dimensional rotations about the z-axis.

$\theta_{11} = \theta_{22} = \phi$, $\theta_{33} = 0$, $\theta_{13} = \theta_{23} = \theta_{31} = \theta_{32} = \pi/2$

$\theta_{12} = \frac{\pi}{2} - \phi$, $\theta_{21} = \frac{\pi}{2} + \phi$



$$\Rightarrow \quad A = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The relation $\vec{r}\,' = A\vec{r}$ has two interpretations. $\left[\vec{r} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right]$

(1) $\vec{r}\,'$ is the representation of the same vector $\vec{r}$ in a rotated frame S'. (Passive view)

(2) $\vec{r}\,'$ is a new vector in the same frame, but rotated in the opposite direction. (Active view)

Infinitesimal rotations.

For infinitesimal rotations,

① $\{\theta_{ll}\}$ are small $\Rightarrow \cos\theta_{ll} \approx 1$ (no summation convention)

② $\{\theta_{lm}\}$ are close to $\pi/2$ for $l \neq m \Rightarrow \cos\theta_{lm} \approx 0$

$\therefore$ An infinitesimal rotation matrix can be written as,

$A = \mathbb{1} + \epsilon$, where $\mathbb{1}$ is the identity matrix and

the elements of the matrix ϵ are small.

i.e. $A_{lm} = \delta_{lm} + \epsilon_{lm}$ where $\epsilon_{lm} \ll 1$.

Infinitesimal rotations commute:

Consider rotations $A_1 = \mathbb{1} + \epsilon_1$ and $A_2 = \mathbb{1} + \epsilon_2$

Suppose A_2 acts before A_1. The net effect is captured by the rotation matrix, $A = A_1 A_2 = (\mathbb{1}+\epsilon_1)(\mathbb{1}+\epsilon_2)$

$$= \mathbb{1} + \epsilon_1 + \epsilon_2 + \epsilon_1\epsilon_2$$

$$\Rightarrow A \approx \mathbb{1} + \epsilon_1 + \epsilon_2$$

Since addition of matrices is commutative, infinitesimal rotations commute.

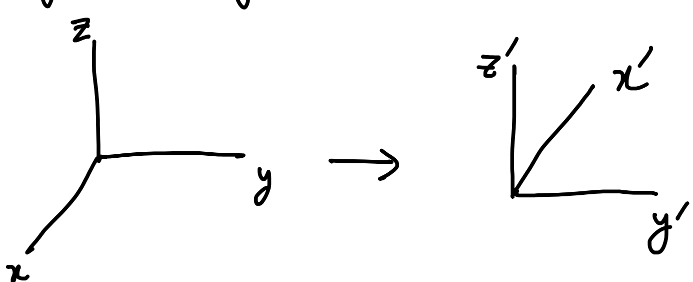
<u>Determinant of an orthogonal matrix</u>

$$AA^T = \mathbb{1} \Rightarrow \det(AA^T) = \det(\mathbb{1}) = 1$$

$$\Rightarrow (\det A)^2 = 1 \Rightarrow \det A = \pm 1$$

An orthogonal matrix whose determinant is -1 represents unphysical transformations.

E.g. the matrix $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ makes a right handed reference frame left handed.



$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ y \\ z \end{pmatrix}$$

The same is true for inversion $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -\mathbb{1}$.

Any orthogonal transformation with determinant -1, can be factorized into an inversion and a physical transformation whose determinant is $+1$.

Alternative argument: The identity operation is a physical transformation and has determinant 1. A small change to it to form a transformation corresponding to an infinitesimal rotation cannot change the determinant from $+1$ to -1. Thus, the determinant of an infinitesimal rotation matrix must be $+1$. A finite rotation can be thought of as being a result of continuous changes to the identity operation. In such a continuous process, the value of the determinant must vary continuously. Since the only allowed values are ± 1, its value must always be $+1$.

A matrix with determinant $+1$ corresponds to a proper transformation and that with determinant -1 corresponds to an improper transformation.