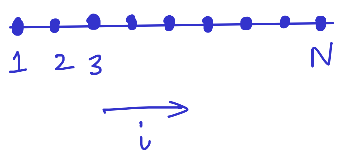
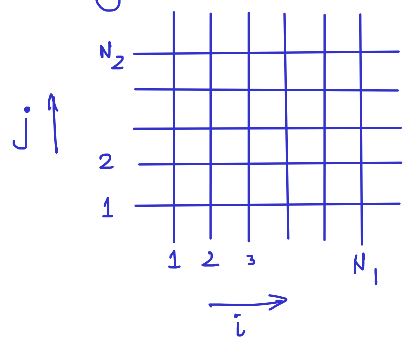
Lagrangian formulation for continuous systems

Applications → Elastic solids,
Sound in gases,
Electromagnetic radiation, ...

Degrees of freedom	1	2	N
Lagrangian	$L(q, \dot{q}, t)$	$L(q_1, q_2, \dot{q}_1, \dot{q}_2, t)$	$L(\{q_i\}, \{\dot{q}_i\}, t)$

Discrete spatially extended system: Lattice systems





$N_1 N_2 = N$

Generalized coordinates → Displacements at Lattice sites (e.g.)

Continuous system: 1D: $N \to \infty$, $i \to x$, $L = \int \mathcal{L}\, dx$,

where $\mathcal{L}\left(\eta, \frac{d\eta}{dx}, \frac{d\eta}{dt}, x, t\right)$ is the Lagrangian density

and η is a continuous **field** variable.

In d-spatial dimensions, $L = \int \mathcal{L}\, d^d x$, and,

$$\mathcal{L} = \mathcal{L}\left(\eta, \frac{\partial \eta}{\partial x_1}, \frac{\partial \eta}{\partial x_2}, \dots, \frac{\partial \eta}{\partial x_d}, \frac{\partial \eta}{\partial t}, x_1, x_2, \dots, x_d, t\right) = \mathcal{L}\left(\eta, \vec{\nabla}\eta, \frac{\partial \eta}{\partial t}, \vec{x}, t\right)$$

There can be more than one field variable. In the example we will consider, there will be three field variables – the displacements along the three Cartesian directions.

$$\mathcal{L} = \mathcal{L}\left(\{\eta_i\}, \{\eta_{i,\nu}\}, x_\nu\right)$$

where $x_\nu \equiv (t, \vec{x})$ i.e., $x_0 = t$, $(x_1, x_2, x_3) = \vec{x}$

$$\eta_{i,\nu} = \frac{\partial \eta_i}{\partial x_\nu}.$$

Later, we will also need second derivatives. $\eta_{i,\mu\nu} \equiv \dfrac{\partial^2 \eta_i}{\partial x_\mu \partial x_\nu}$

Read on your own (not in syllabus): Discrete to continuous formulation of a one-dimensional elastic solid.

Acoustic field in gases

Consider a gas occupying an equilibrium volume V_0 and having mass M. Its equilibrium density is $\mu_0 \equiv M/V_0$.

Let the equilibrium pressure be P_0.

The Lagrangian for this system is $L = \int \mathcal{L}\, dV$, $\mathcal{L}$ = Lagrangian density.

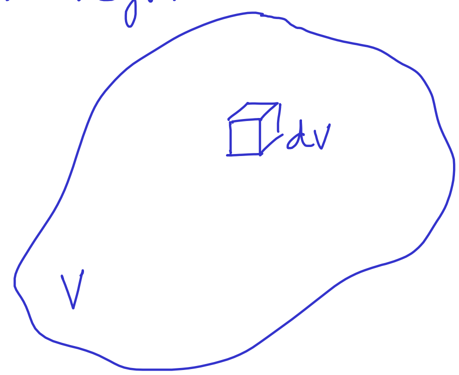
In this system we can define a potential energy and $L = K.E. - P.E.$

$\therefore \quad \mathcal{L} = \mathcal{T} - \mathcal{V}$, where $\mathcal{T}$ = kinetic energy density and

$\mathcal{V}$ = Potential energy density.

If $\vec{\eta}$ represents displacement of gas molecules in a region dV around the location $\vec{x}$, then,

$$\mathcal{T} = \frac{1}{2}\mu_0 \dot{\eta}^2 , \qquad \dot{\eta}^2 = \dot{\vec{\eta}} \cdot \dot{\vec{\eta}}$$



Assume that the gas we consider occupies a small enough volume so that the potential energy density is constant in the volume, i.e.,

Potential energy of the gas is $\mathcal{V} V_0$.

The work done on the gas in changing its volume by dV is $-P\,dV$.

$\therefore$ Potential Energy of the gas when its volume is $V = V_0 + \Delta V$ is given by

$$\mathcal{V} V_0 = - \int_{V_0}^{V_0+\Delta V} P \, dV$$

Now, $\quad P = P_0 + \left(\frac{\partial P}{\partial V}\right)_0 \Delta V$

$$\Rightarrow \mathcal{V} V_0 = -P_0 \Delta V - \frac{1}{2}\left(\frac{\partial P}{\partial V}\right)_0 (\Delta V)^2$$

$$\Rightarrow \mathcal{V} = -P_0\left(\frac{\Delta V}{V_0}\right) - \frac{1}{2}\left(\frac{\partial P}{\partial V}\right)_0 \frac{(\Delta V)^2}{V_0}$$

The vibrations of sound in the gas are so fast that it can be assumed that the gas vibrates adiabatically.

$$\Rightarrow \quad PV^\gamma = \text{constant}$$

$$\Rightarrow \quad V^\gamma\left(\frac{\partial P}{\partial V}\right) + P\gamma V^{\gamma-1} = 0 \Rightarrow \left(\frac{\partial P}{\partial V}\right)_0 = -\frac{\gamma P_0}{V_0}$$

The density is $\mu = M/V$. This has small fluctuations around the equilibrium density $\mu_0 = M/V_0$.

$$\Rightarrow \ln\mu = \ln M - \ln V$$

$$\Rightarrow \frac{\Delta\mu}{\mu_0} = -\frac{\Delta V}{V_0}$$

The fractional change in density is denoted by σ, i.e., $\sigma = \frac{\Delta\mu}{\mu_0}$.

$$\therefore \mathcal{V} = \sigma P_0 + \frac{\gamma P_0}{2}\sigma^2$$

Next, we will express σ in terms of $\vec{\eta}$ and its derivatives.

Consider a volume V in space.

Mass flowing out $= \mu_0 \int \vec{\eta}\cdot\vec{dS}$

Mass increase due to density change is $\int(\sigma\mu_0)\,dV$.

These must add up to zero.

$$\therefore \quad \mu_0\int\vec{\eta}\cdot\vec{dS} + \int(\sigma\mu_0)\,dV = 0$$

$$\Rightarrow \mu_0\int(\vec{\nabla}\cdot\vec{\eta} + \sigma)\,dV = 0$$

Since this must hold for every region, $\sigma = -\vec{\nabla}\cdot\vec{\eta}$

$$\Rightarrow \mathcal{V} = -P_0\vec{\nabla}\cdot\vec{\eta} + \frac{\gamma P_0}{2}(\vec{\nabla}\cdot\vec{\eta})^2$$

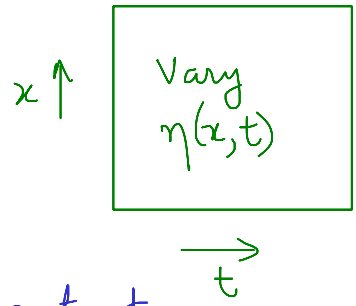
The first term cannot contribute to the total potential energy (Check).

$$\therefore \quad \boxed{\mathcal{L} = \frac{1}{2}\left[\mu_0 \dot{\vec{\eta}}^2 + 2P_0 \vec{\nabla}\cdot\vec{\eta} - \gamma P_0 \left(\vec{\nabla}\cdot\vec{\eta}\right)^2\right]}$$

The Euler-Lagrange equations

Consider the one dimensional problem

$$L = \int \mathcal{L}\, dx$$



The action is, $I = \int L\, dt = \int \mathcal{L}\, dx\, dt$.

The field variables are varied for all (x,t) except at the boundaries of the domain, using a parameter α. The physical equations of motion are obtained by extremizing I.

$$\mathcal{L} = \mathcal{L}\left(\eta, \frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial t}, x, t\right)$$

$$\eta(x,t,\alpha) = \underbrace{\eta(x,t)}_{\text{physical value of the field}} + \alpha \zeta(x,t)$$

$$\frac{dI}{d\alpha} = \int_{x_1}^{x_2}\int_{t_1}^{t_2} dx\,dt \left[\left(\frac{\partial \mathcal{L}}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial \alpha}\right) + \left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial t}\right)}\right)\left(\frac{\partial\left(\frac{\partial \eta}{\partial t}\right)}{\partial \alpha}\right) + \left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial x}\right)}\right)\left(\frac{\partial\left(\frac{\partial \eta}{\partial x}\right)}{\partial \alpha}\right)\right]$$

$$\int_{t_1}^{t_2} dt \left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial t}\right)}\right)\left(\frac{\partial\left(\frac{\partial \eta}{\partial t}\right)}{\partial \alpha}\right) = \int dt \left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial t}\right)}\right)\frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial \alpha}\right)$$

$$= \underbrace{\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial t}\right)}\right)\left(\frac{\partial \eta}{\partial \alpha}\right)\Bigg|_{t_1}^{t_2}}_{\text{Vanishes since there is no variation at the boundaries}} - \int dt \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial t}\right)}\right)\right]\left(\frac{\partial \eta}{\partial \alpha}\right) = -\int dt \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial t}\right)}\right)\right]\left(\frac{\partial \eta}{\partial \alpha}\right)$$

Similarly, $\displaystyle\int_{x_1}^{x_2} dx \left(\frac{\partial \mathcal{L}}{\left(\frac{\partial \eta}{\partial x}\right)}\right)\left(\frac{\partial\left(\frac{\partial \eta}{\partial x}\right)}{\partial \alpha}\right) = -\int dx \left[\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\left(\frac{\partial \eta}{\partial x}\right)}\right)\right]\left(\frac{\partial \eta}{\partial \alpha}\right)$

$$\therefore \frac{dI}{d\alpha} = \int dx\, dt \left(\frac{d\eta}{d\alpha}\right)\left[\frac{\partial \mathcal{L}}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\left(\frac{\partial \eta}{\partial t}\right)}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\left(\frac{\partial \eta}{\partial x}\right)}\right)\right] = 0$$

Since the variations are arbitrary, therefore,

$$\boxed{\frac{\partial \mathcal{L}}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\left(\frac{\partial \eta}{\partial t}\right)}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\left(\frac{\partial \eta}{\partial x}\right)}\right) = 0}$$

In d-spatial dimensions, with p, fields, this generalizes to,

$$\boxed{\frac{\partial \mathcal{L}}{\partial \eta_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\left(\frac{\partial \eta_i}{\partial t}\right)}\right) - \sum_{j=1}^{d} \frac{d}{dx_j}\left(\frac{\partial \mathcal{L}}{\left(\frac{\partial \eta}{\partial x_j}\right)}\right) = 0} \quad \text{for } i = 1, 2, \ldots, p$$

This can be compactly written as,

$$\frac{\partial \mathcal{L}}{\partial \eta_i} - \sum_{\nu=0}^{d} \frac{d}{dx_\nu}\left(\frac{\partial \mathcal{L}}{\partial \eta_{i,\nu}}\right) = 0, \quad \text{where } x_0 = t,\ (x_1, x_2, x_3) = \vec{x} \text{ and } \eta_{i,\nu} = \frac{\partial \eta_i}{\partial x_\nu}$$

For $\mathcal{L} = \frac{1}{2}\left[\mu_0 \dot{\eta}^2 + 2 P_0 \vec{\nabla}\cdot\vec{\eta} - \gamma P_0 \left(\vec{\nabla}\cdot\vec{\eta}\right)^2\right]$, let's calculate the Euler-Lagrange equations.

Here $\dfrac{\partial \mathcal{L}}{\partial \eta_i} = 0$, $\dfrac{\partial \mathcal{L}}{\partial \eta_{i,0}} = \dfrac{\partial \mathcal{L}}{\partial \dot{\eta}_i} = \mu_0 \dot{\eta}_i = \mu_0 \eta_{i,0} \Rightarrow \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \eta_{i,0}}\right) = \mu_0 \eta_{i,00}$

For $j = 1, 2, 3$, $\dfrac{\partial \mathcal{L}}{\partial \eta_{i,j}} = P_0 - \gamma P_0 \left(\vec{\nabla}\cdot\vec{\eta}\right) \Rightarrow \dfrac{d}{dx_i}\left(\dfrac{\partial \mathcal{L}}{\partial \eta_{i,j}}\right) = -\gamma P_0 \dfrac{\partial}{\partial x_i}\left(\vec{\nabla}\cdot\vec{\eta}\right)$

$\therefore$ We have,

$$\mu_0 \eta_{i,00} - \gamma P_0 \frac{\partial}{\partial x_i}\left(\vec{\nabla}\cdot\vec{\eta}\right) = 0$$

$$\Rightarrow \mu_0 \frac{\partial^2 \vec{\eta}}{\partial t^2} = \gamma P_0 \vec{\nabla}\left(\vec{\nabla}\cdot\vec{\eta}\right)$$

Taking divergence, and using $\sigma = -\vec{\nabla}\cdot\vec{\eta}$, we have,

$$\mu_0 \frac{\partial^2 \sigma}{\partial t^2} = \gamma P_0 \nabla^2 \sigma$$

$$\Rightarrow \quad \frac{\partial^2 \sigma}{\partial t^2} = v^2 \nabla^2 \sigma \qquad \text{where } v = \sqrt{\frac{\gamma P_0}{\mu_0}} .$$

Thus, the fractional change in density satisfies the wave equation, with sound speed, v, given by $v = \sqrt{\frac{\gamma P_0}{\mu_0}}$.

Comments about the Lagrangian density for the acoustic field in gases:

① The Lagrangian density $\mathcal{L} = \frac{1}{2}\left(\rho_0 \dot{\sigma}^2 - \gamma P_0 (\nabla \sigma)^2\right)$ also results in the same wave equation for σ, but does not have the same physical content.

② The term proportional to $(\vec{\nabla}.\vec{\eta})$ does not affect the equations of motion. This is analogous to the fact that adding a total time derivative of an arbitrary function of generalized coordinates and time, does not affect equations of motion for a discrete system. For continuous systems, terms of the form $\sum_\nu \frac{\partial}{\partial x_\nu} F_\nu(\vec{\eta}, x_\nu)$ in the Lagrangian do not affect equations of motion.

From now on the Einstein summation convention will be used.

i.e. $a_i b_i \equiv \sum_{i=1}^{d} a_i b_i$ and $a_\nu b_\nu \equiv \sum_{\nu=0}^{d} a_\nu b_\nu$

Covariant / contravariant notations will not be necessary and will not be used.

The Lagrangian formulation had spacetime symmetry. For a single field η,

$\mathcal{L} = \mathcal{L}(\eta, \{\eta_{,\nu}\}, \{x_\nu\})$ $\qquad \eta_{,\nu} = \frac{\partial \eta}{\partial x_\nu}$, $\nu = 0, 1, 2, \ldots, d$

The Euler-Lagrange equations: $\frac{d}{dx_\nu}\left(\frac{\partial \mathcal{L}}{\partial \eta_{,\nu}}\right) = \frac{\partial \mathcal{L}}{\partial \eta}$

Hamiltonian formulation for continuous systems

In discrete systems, starting from a Lagrangian description with $L(\{q_i\}, \{\dot{q}_i\}, t)$, we could go to a Hamiltonian description, where,

$$H = \sum_i p_i \dot{q}_i - L \quad \text{where } p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

The Hamiltonian function depends on $(\{q_i\}, \{p_i\}, t)$ and the dependence on the generalized velocities has been suppressed.

For continuous systems, $\sum_i$ is replaced by $\int d^d V$. H and L are written as integrals over corresponding densities and we need to write the momenta in terms of the corresponding densities.

$$\sum_i p_i \dot{q}_i \rightarrow \int \underbrace{d^d x\, \pi}_{\text{momentum}} \dot{\eta} \qquad \pi = \text{momentum density}$$

$$\mathcal{H} = \pi \dot{\eta} - \mathcal{L} \,, \qquad \pi = \frac{\partial \mathcal{L}}{\partial \dot{\eta}}$$

$\mathcal{L}$ is a function of $(\eta, \eta_{,\nu}, x_\nu)$. In $\mathcal{H}$, dependence on $\dot{\eta}$ is replaced by dependence on π.

$$\therefore \quad \mathcal{H} = \mathcal{H}(\eta, \underbrace{\eta_{,i}}_{\vec{\nabla}\eta}, \pi, x_\nu) \qquad \text{* No Spacetime symmetry *}$$

[In our discussion, we will talk about a single field. When there are more fields,

$$\mathcal{H} = \sum_i \pi_i \dot{\eta}_i - \mathcal{L} \quad \text{where} \quad \pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\eta}_i}.$$

$$\mathcal{H} = \mathcal{H}(\{\eta_i\}, \{\eta_{i,j}\}, \pi_i, x_\nu), \qquad j = 1, 2, \ldots, d \;]$$

Derivation of Hamilton's equations of motion

$$\mathcal{H} = \pi \dot{\eta} - \mathcal{L}$$

$$\Rightarrow \frac{\partial \mathcal{H}}{\partial \pi} = \dot{\eta} + \pi \frac{\partial \dot{\eta}}{\partial \pi} - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\eta}}}_{\pi} \frac{\partial \dot{\eta}}{\partial \pi} = \dot{\eta} \qquad [\because \eta \text{ and } \pi \text{ are independent}]$$

[Note: The quantities $\eta, \{\eta_{,\nu}\}$ are mutually independent, from the Lagrangian description. The quantities $\eta, \pi, \{\eta_{,i}\}$ are mutually independent, from the Hamiltonian description. We are now using the Hamiltonian description]

$$\frac{\partial \mathcal{H}}{\partial \eta} = \pi \frac{\partial \dot{\eta}}{\partial \eta} - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\eta}}}_{\pi} \frac{\partial \dot{\eta}}{\partial \eta} - \frac{\partial \mathcal{L}}{\partial \eta} = -\frac{\partial \mathcal{L}}{\partial \eta} = -\frac{d}{dx_\mu}\left(\frac{\partial \mathcal{L}}{\partial \eta_{,\mu}}\right) \qquad \text{(using Euler-Lagrange equations)}$$

$$\Rightarrow \frac{\partial \mathcal{H}}{\partial \eta} = -\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}}\right) - \frac{d}{dx_i}\left(\frac{\partial \mathcal{L}}{\partial \eta_{,i}}\right)$$

Now, $\displaystyle \frac{\partial \mathcal{H}}{\partial \eta_{,i}} = \pi \frac{\partial \dot{\eta}}{\partial \eta_{,i}} - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\eta}}}_{\pi} \frac{\partial \dot{\eta}}{\partial \eta_{,i}} - \frac{\partial \mathcal{L}}{\partial \eta_{,i}} = -\frac{\partial \mathcal{L}}{\partial \eta_{,i}}$

$$\Rightarrow \frac{\partial \mathcal{H}}{\partial \eta} = -\dot{\pi} + \frac{d}{dx_i}\left(\frac{\partial \mathcal{H}}{\partial \eta_{,i}}\right)$$

Thus, our Hamilton's equations take the following form.

$$\boxed{\dot{\eta} = \frac{\partial \mathcal{H}}{\partial \pi}, \text{ and, } \dot{\pi} = -\frac{\partial \mathcal{H}}{\partial \eta} + \frac{d}{dx_i}\left(\frac{\partial \mathcal{H}}{\partial \eta_{,i}}\right)}$$

For a field ψ, we define the functional derivative as follows.

$$\frac{\delta}{\delta \psi} \equiv \frac{\partial}{\partial \psi} - \frac{d}{dx_i}\frac{\partial}{\partial \psi_{,i}}$$

$$\frac{\delta \mathcal{H}}{\delta \pi} = \frac{\partial \mathcal{H}}{\partial \pi} - \frac{d}{dx_i}\underbrace{\frac{\partial \mathcal{H}}{\partial \pi_{,i}}}_{\substack{0 \text{ because} \\ \mathcal{H} \text{ is independent} \\ \text{of } \pi_{,i}}} = \frac{\partial \mathcal{H}}{\partial \pi}$$

$$\therefore \boxed{\dot{\eta} = \frac{\delta \mathcal{H}}{\delta \pi}, \text{ and, } \dot{\pi} = -\frac{\delta \mathcal{H}}{\delta \eta}}$$

In this notation, the Euler-Lagrange equation becomes, $\displaystyle \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}}\right) = \frac{\delta \mathcal{L}}{\delta \eta}$.

$$\mathcal{H} = \mathcal{H}(\eta, \pi, \{\eta_{,i}\}, \{x_\nu\})$$

$$\frac{d\mathcal{H}}{dt} = \underbrace{\frac{\partial \mathcal{H}}{\partial \eta}\dot{\eta}}_{①} + \underbrace{\underbrace{\frac{\partial \mathcal{H}}{\partial \pi}}_{\dot{\eta}}\dot{\pi}}_{②} + \underbrace{\frac{\partial \mathcal{H}}{\partial \eta_{,i}}\dot{\eta}_{,i}}_{③} + \frac{\partial \mathcal{H}}{\partial t} \qquad \text{—(1)}$$

Using $\mathcal{H} = \pi\dot{\eta} - \mathcal{L}$, we get,

$$\frac{d\mathcal{H}}{dt} = \overset{②}{\dot{\pi}\dot{\eta}} + \overset{④}{\pi\ddot{\eta}} - \overset{①}{\underbrace{\frac{\partial\mathcal{L}}{\partial\eta}}_{-\frac{\partial\mathcal{H}}{\partial\eta}}\dot{\eta}} - \overset{④}{\underbrace{\frac{\partial\mathcal{L}}{\partial\dot{\eta}}}_{\pi}\ddot{\eta}} - \overset{③}{\underbrace{\frac{\partial\mathcal{L}}{\partial\eta_{,i}}}_{-\frac{\partial\mathcal{H}}{\partial\eta_{,i}}}\dot{\eta}_{,i}} - \frac{\partial\mathcal{L}}{\partial t} \quad -(2)$$

Comparing Eqs. (1) and (2), we get, $\boxed{\frac{\partial\mathcal{H}}{\partial t} = -\frac{\partial\mathcal{L}}{\partial t}}$

From Eq. (1), we get,

$$\frac{d\mathcal{H}}{dt} = \underbrace{\left[-\dot{\pi} + \frac{d}{dx_i}\left(\frac{\partial\mathcal{H}}{\partial\eta_{,i}}\right)\right]}_{\frac{\partial\mathcal{H}}{\partial\eta}}\dot{\eta} + \dot{\eta}\dot{\pi} + \frac{\partial\mathcal{H}}{\partial\eta_{,i}}\frac{d\dot{\eta}}{dx_i} + \frac{\partial\mathcal{H}}{\partial t}$$

$$\Rightarrow \frac{d\mathcal{H}}{dt} = \frac{d}{dx_i}\left[\dot{\eta}\left(\frac{\partial\mathcal{H}}{\partial\eta_{,i}}\right)\right] + \frac{\partial\mathcal{H}}{\partial t}$$

If the Hamiltonian density does not have explicit time dependence, the total energy, $H = \int\mathcal{H}\,dV$, is conserved. [Use divergence theorem.]

Suppose $\mathcal{U}(\eta, \pi, \{\eta_{,\nu}\}, \{\pi_{,\nu}\}, \{x_\nu\})$ is a function of coordinates, momenta and their derivatives.

If $\mathcal{U}$ is a density of some quantity U, i.e., $U = \int\mathcal{U}\,dV$, then,

$$\frac{dU}{dt} = \int dV\left(\frac{d\mathcal{U}}{dt}\right) = \int dV\left[\left(\frac{\partial\mathcal{U}}{\partial\eta}\right)\dot{\eta} + \left(\frac{\partial\mathcal{U}}{\partial\pi}\right)\dot{\pi} + \left(\frac{\partial\mathcal{U}}{\partial\eta_{,\nu}}\right)\dot{\eta}_{,\nu} + \left(\frac{\partial\mathcal{U}}{\partial\pi_{,\nu}}\right)\dot{\pi}_{,\nu} + \frac{\partial\mathcal{U}}{\partial t}\right]$$

$$\int dV\left(\frac{\partial\mathcal{U}}{\partial\eta_{,i}}\right)\dot{\eta}_{,i} = \int dV\left(\frac{\partial\mathcal{U}}{\partial\eta_{,i}}\right)\dot{\eta}_{,i} = \int dV\left(\frac{\partial\mathcal{U}}{\partial\eta_{,i}}\right)\frac{\partial\dot{\eta}}{\partial x_i} = \left.\frac{\partial\mathcal{U}}{\partial\eta_{,i}}\dot{\eta}\right|_{\text{boundary}} - \int dV\frac{d}{dx_i}\left(\frac{\partial\mathcal{U}}{\partial\eta_{,i}}\right)\dot{\eta}$$

$$= -\int dV\frac{d}{dx_i}\left(\frac{\partial\mathcal{U}}{\partial\eta_{,i}}\right)\dot{\eta}$$

Similarly, $\int dV \left(\frac{\partial \mathcal{U}}{\partial \pi_{,i}}\right)\dot{\pi}_{,i} = -\int dV \frac{d}{dx_i}\left(\frac{\partial \mathcal{U}}{\partial \pi_{,i}}\right)\dot{\pi}$

$$\frac{dU}{dt} = \int dV \Bigg[\underbrace{\left\{\frac{\partial \mathcal{U}}{\partial \eta} - \frac{d}{dx_i}\left(\frac{\partial \mathcal{U}}{\partial \eta_{,i}}\right)\right\}}_{\frac{\delta \mathcal{U}}{\delta \eta}} \underbrace{\dot{\eta}}_{\frac{\delta \mathcal{H}}{\delta \pi}} + \underbrace{\left\{\frac{\partial \mathcal{U}}{\partial \pi} - \frac{d}{dx_i}\left(\frac{\partial \mathcal{U}}{\partial \pi_{,i}}\right)\right\}}_{\frac{\delta \mathcal{U}}{\delta \pi}} \underbrace{\dot{\pi}}_{-\frac{\delta \mathcal{H}}{\delta \eta}} \Bigg] + \int dV \frac{\partial \mathcal{U}}{\partial t}$$

$$= \int dV \left[\frac{\delta \mathcal{U}}{\delta \eta}\frac{\delta \mathcal{H}}{\delta \pi} - \frac{\delta \mathcal{U}}{\delta \pi}\frac{\delta \mathcal{H}}{\delta \eta}\right] + \int dV \frac{\partial \mathcal{U}}{\partial t}$$

$$= [U, H] + \frac{\partial U}{\partial t}$$

where $[A, B] \equiv \int dV \left[\frac{\delta \mathcal{A}}{\delta \eta}\frac{\delta \mathcal{B}}{\delta \pi} - \frac{\delta \mathcal{A}}{\delta \pi}\frac{\delta \mathcal{B}}{\delta \eta}\right]$ is the Poisson bracket between A and B.

and $\frac{\partial A}{\partial t} = \int dV \frac{\partial \mathcal{A}}{\partial t}$

Since $[H, H] = 0$ (Check). $\frac{dH}{dt} = \frac{\partial H}{\partial t}$

Some Points to note:

→ Continuum Poisson brackets are defined using densities. Analogies with discrete Poisson brackets are not very useful/elegant.

→ Quantization: Unlike in discrete systems, continuum Poisson brackets cannot be replaced by corresponding commutators to get a quantum theory.

Discrete and continuum description for the same system

Suppose we have fields $\eta(\vec{r}, t)$ and their corresponding momentum densities $\pi(\vec{r}, t)$ defined in a hypercubic volume $V = L^d$.

In the Fourier space we have,

$$q_{\vec{k}}(t) \equiv \frac{1}{\sqrt{V}} \int \eta(\vec{r}, t)\, e^{-i\vec{k}.\vec{r}}\, dV \quad ,$$

where the components of $\vec{k}$ are integer multiples of $2\pi/L$.

$$p_{\vec{k}}(t) \equiv \frac{1}{\sqrt{V}} \int \pi(\vec{r},t)\, e^{i\vec{k}\cdot\vec{r}}\, dV$$

[This convention is used so that our final expressions look nice.]

Inverting, we have,

$$\eta(\vec{r},t) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} q_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}}, \text{ and, } \quad \pi(\vec{r},t) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} p_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{r}}$$

$\therefore$ $q_{\vec{k}}$ is a quantity whose density is $\frac{1}{\sqrt{V}} \eta(\vec{r},t)\, e^{-i\vec{k}\cdot\vec{r}}$ and $p_{\vec{k}}$ has density $\frac{1}{\sqrt{V}} \pi(\vec{r},t)\, e^{i\vec{k}\cdot\vec{r}}$

$$\therefore \left[q_{\vec{k}}, p_{\vec{k}'}\right] = \frac{1}{V} \int dV\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}} \Big(\underbrace{\frac{\delta\eta}{\delta\eta}\frac{\delta\pi}{\delta\pi}}_{1} - \underbrace{\frac{\delta\eta}{\delta\pi}\frac{\delta\pi}{\delta\eta}}_{0} \Big) = \delta_{\vec{k},\vec{k}'}$$

Similarly, $\left[q_{\vec{k}}, q_{\vec{k}'}\right] = \left[p_{\vec{k}}, p_{\vec{k}'}\right] = 0$

$$\dot{q}_{\vec{k}} = \left[q_{\vec{k}}, H\right] = \frac{1}{\sqrt{V}} \int dV\, e^{-i\vec{k}\cdot\vec{r}} \left(\frac{\delta\eta}{\delta\eta}\frac{\delta\mathcal{H}}{\delta\pi} - \frac{\delta\eta}{\delta\pi}\frac{\delta\mathcal{H}}{\delta\eta} \right)$$

$$= \frac{1}{\sqrt{V}} \int dV\, e^{-i\vec{k}\cdot\vec{r}} \frac{\delta\mathcal{H}}{\delta\pi}$$

$$\text{Now, } \frac{\partial H}{\partial p_{\vec{k}}} = \int dV \frac{\partial\mathcal{H}}{\partial p_{\vec{k}}} = \int dV \frac{\partial\mathcal{H}}{\partial\pi}\frac{\partial\pi}{\partial p_{\vec{k}}} = \frac{1}{\sqrt{V}} \int dV \underbrace{\frac{\partial\mathcal{H}}{\partial\pi}}_{=\frac{\delta\mathcal{H}}{\delta\pi}} e^{-i\vec{k}\cdot\vec{r}} = \dot{q}_{\vec{k}}$$

$$\dot{p}_{\vec{k}} = \left[p_{\vec{k}}, H\right] = \frac{1}{\sqrt{V}} \int dV\, e^{i\vec{k}\cdot\vec{r}} \left(\frac{\delta\pi}{\delta\eta}\frac{\delta\mathcal{H}}{\delta\pi} - \frac{\delta\pi}{\delta\pi}\frac{\delta\mathcal{H}}{\delta\eta} \right) = -\frac{1}{\sqrt{V}} \int dV\, e^{i\vec{k}\cdot\vec{r}} \frac{\delta\mathcal{H}}{\delta\eta}$$

$$\frac{\partial H}{\partial q_{\vec{k}}} = \int dV \left[\frac{\partial\mathcal{H}}{\partial\eta}\frac{\partial\eta}{\partial q_{\vec{k}}} + \frac{\partial\mathcal{H}}{\partial\eta_{,i}}\frac{\partial\eta_{,i}}{\partial q_{\vec{k}}} \right] = \int dV \left[\frac{\partial\mathcal{H}}{\partial\eta} - \frac{d}{dx_i}\left(\frac{\partial\mathcal{H}}{\partial\eta_{,i}}\right) \right] \frac{\partial\eta}{\partial q_{\vec{k}}}$$

(Using integration by parts and setting the surface term to zero.]

$$\Rightarrow \quad \frac{\partial H}{\partial q_{\vec{k}}} = \frac{1}{\sqrt{V}} \int dV \, \frac{\delta \mathcal{H}}{\delta \eta} \, e^{i\vec{k}\cdot\vec{r}} = \dot{p}_{\vec{k}}$$

Thus, $\{p_{\vec{k}}\}$ and $\{q_{\vec{k}}\}$ satisfy Hamilton's equations for a discrete system with $H = \int \mathcal{H} \, dV$.