The Logistic Equation

Applications — Population dynamics, Predator-prey models

$$\boxed{x_{n+1} = a x_n (1 - x_n), \qquad 0 \leq x_n \leq 1, \qquad a > 1}$$

Fixed points: $\quad x_\infty = a x_\infty (1 - x_\infty)$

Trivial fixed point $\rightarrow$ $x_\infty = 0$

Non trivial fixed point $\rightarrow$ $1 = a(1 - x_\infty) \Rightarrow x_\infty = 1 - 1/a$

Stability of the fixed points

$x_\infty = 0$:

Let $x_n = \delta \qquad \delta \ll 1$

$$x_{n+1} = a\delta(1-\delta) \approx a\delta$$

$\therefore$ x_n approaches towards $x_\infty = 0$ if $|a| < 1$.

$\Rightarrow$ It is unstable if $a > 1$

$x_\infty = 1 - 1/a$:

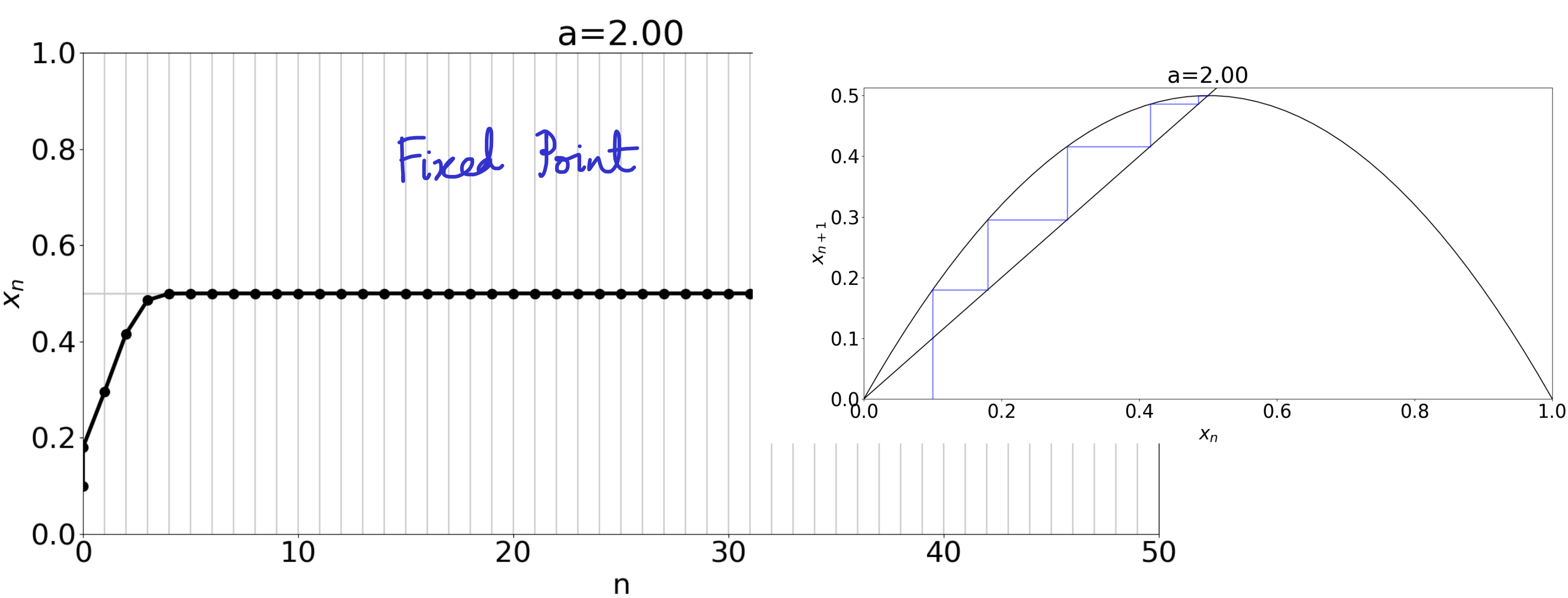
Let $x_n = 1 - 1/a + \delta$

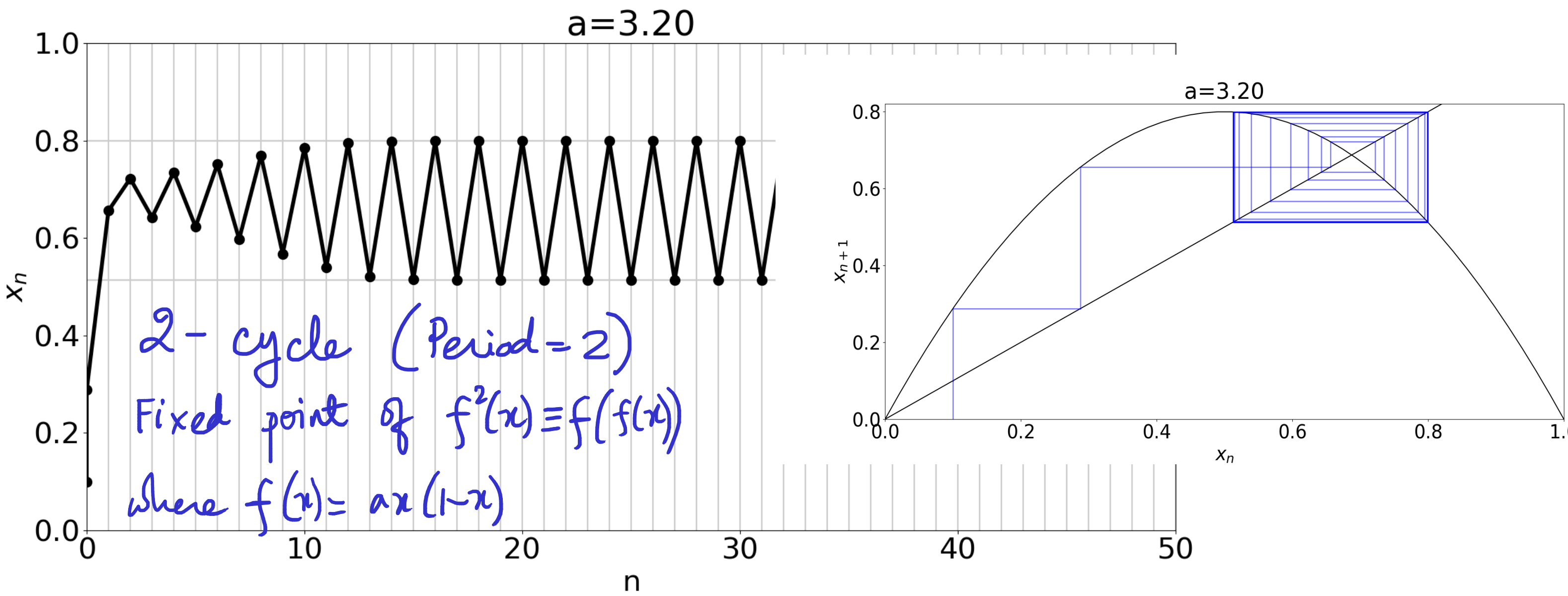
$$x_{n+1} = a(1 - 1/a + \delta)(1/a - \delta)$$

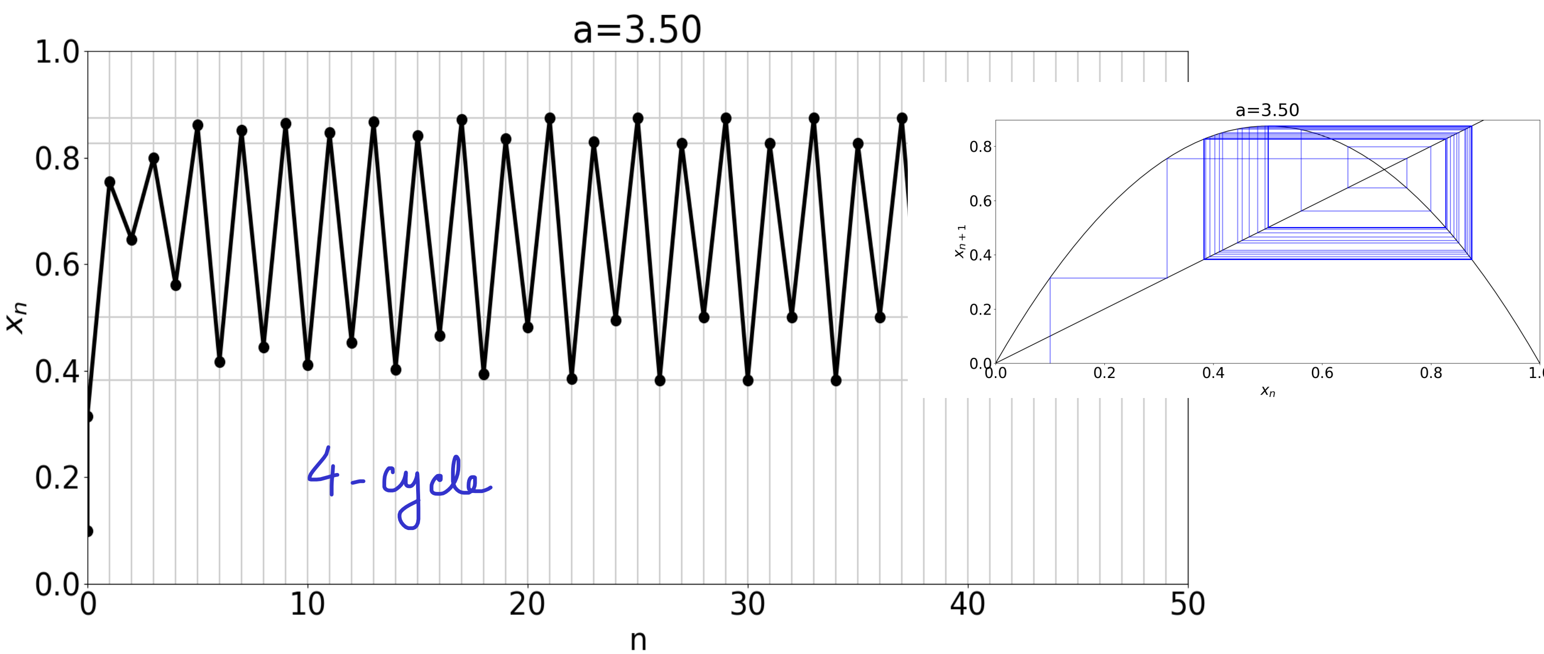
$$= 1 - 1/a + (2-a)\delta$$

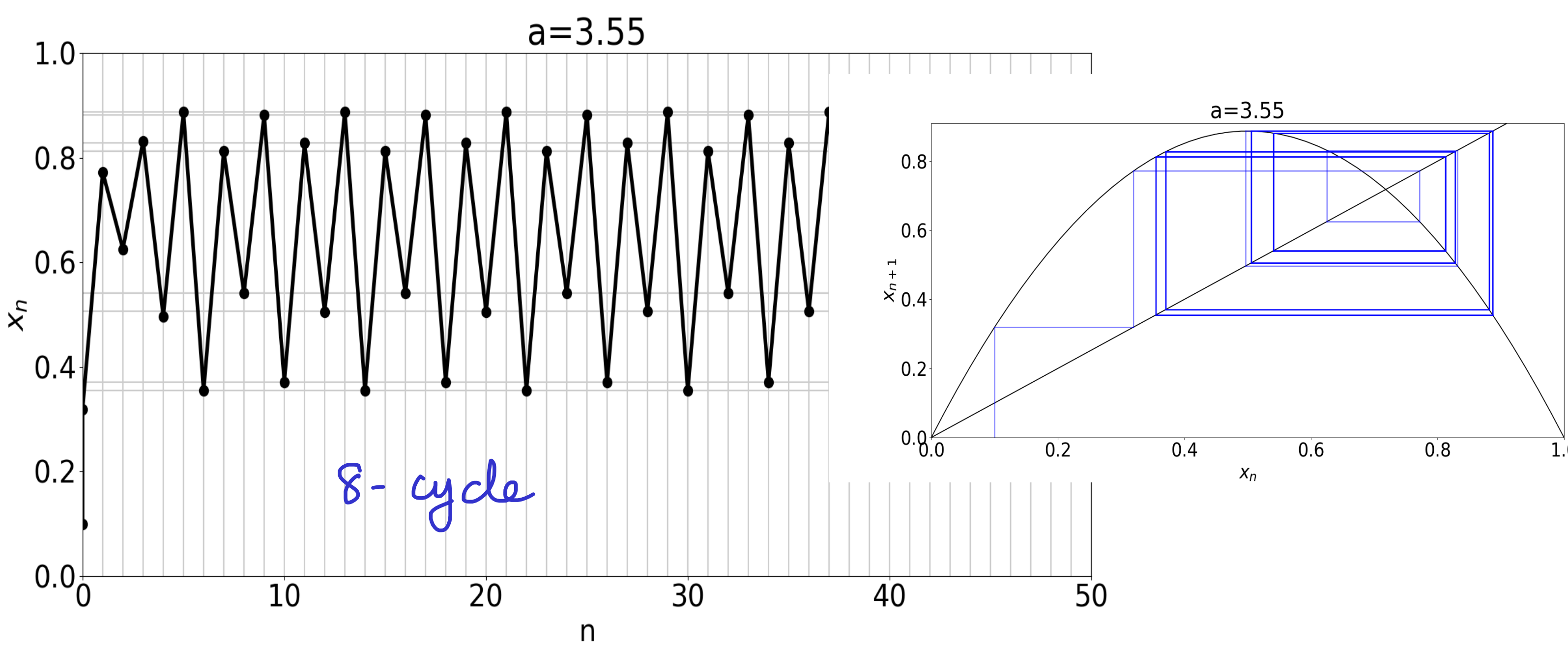
$\Rightarrow$ x_n approaches $(1 - 1/a)$ if $|2-a| < 1 \Rightarrow -1 < 2-a < 1$

$\Rightarrow$ $1 < a < 3$

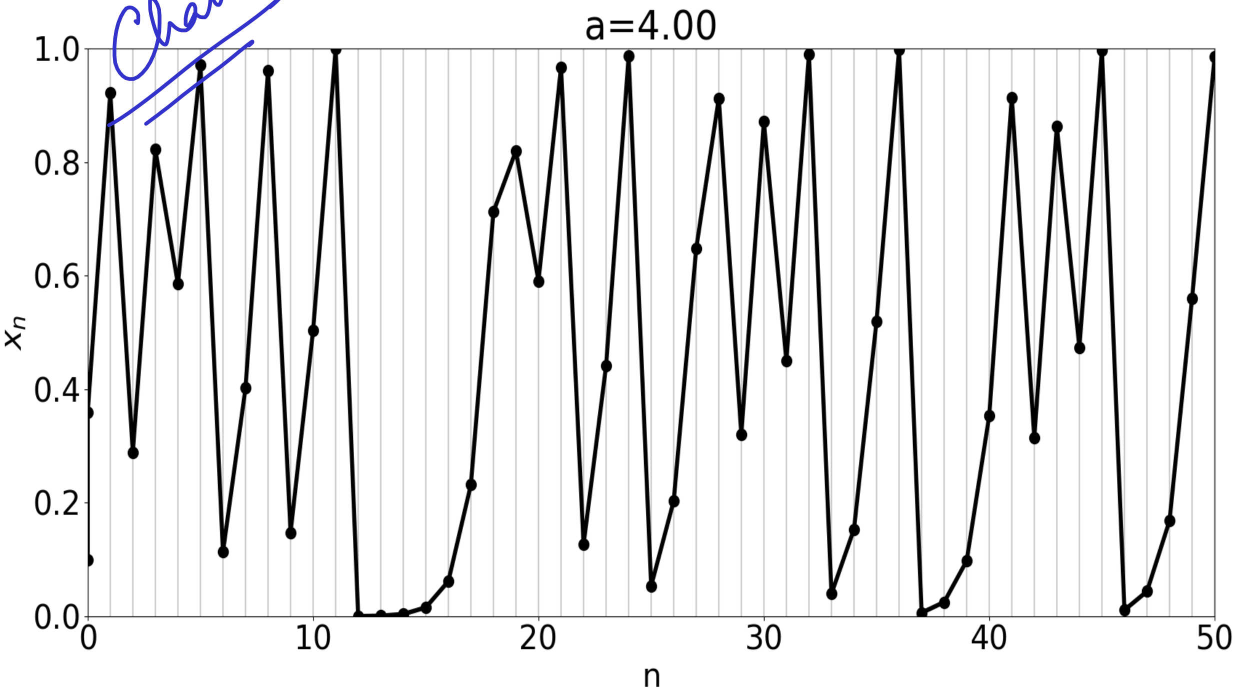



a=3.20
2-cycle (Period = 2)
Fixed point of $f^2(x) \equiv f(f(x))$
where $f(x) = ax(1-x)$
n
x_n
x_{n+1}

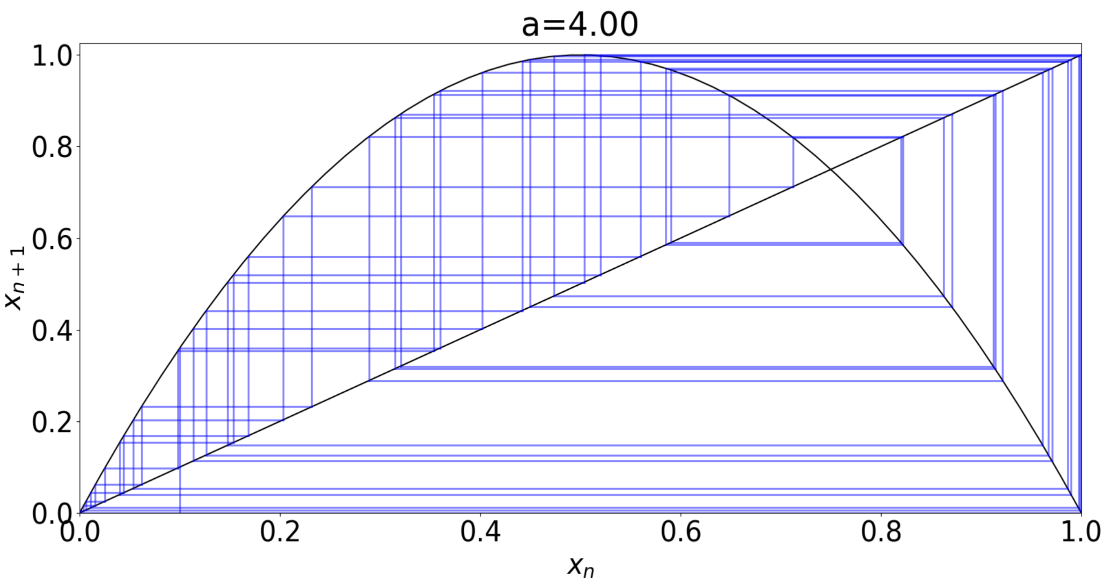

a=3.50
4-cycle
n
x_n
x_{n+1}


a=3.55
8-cycle
n
x_n
x_{n+1}

Chaos

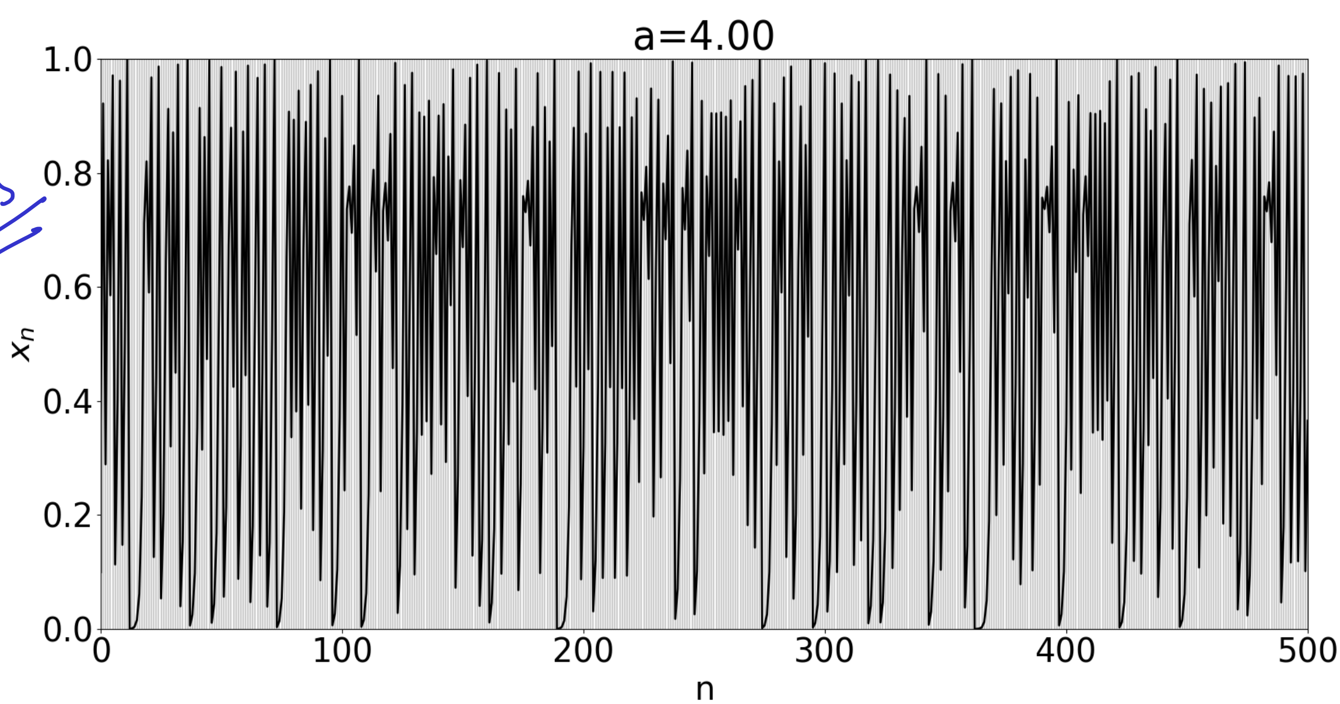


a=4.00
1.0
0.8
0.6
0.4
0.2
0.0
x_n
0
10
20
30
40
50
n

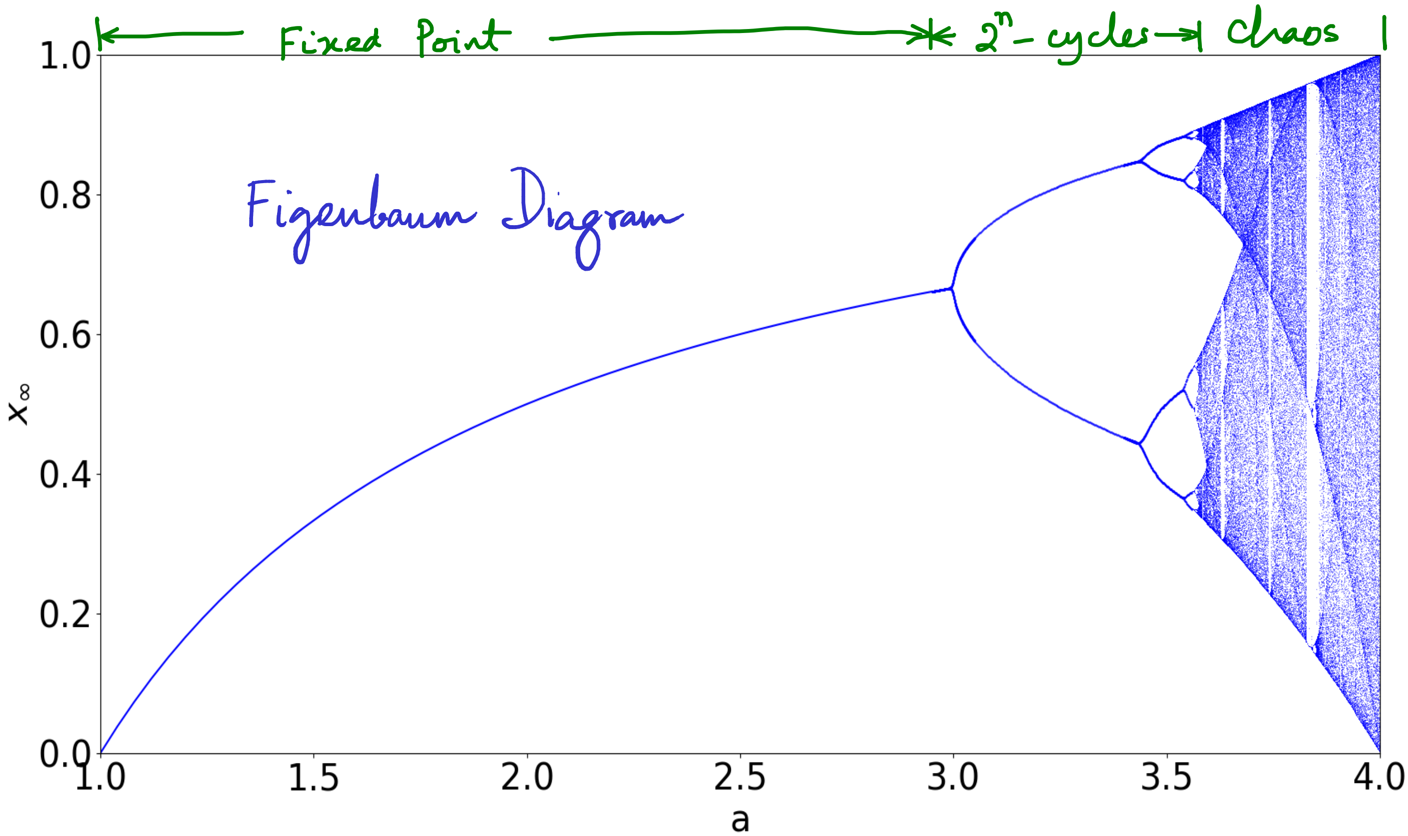


a=4.00
1.0
0.8
0.6
0.4
0.2
0.0
x_{n+1}
0.0
0.2
0.4
0.6
0.8
1.0
x_n

Chaos



a=4.00
1.0
0.8
0.6
0.4
0.2
0.0
x_n
0
100
200
300
400
500
n



- For $a = 4$, after 10-11 iterations, we get a sequence of numbers which appear random.

- The bifurcations (2^n to 2^{n+1} cycle) look similar to each other when enlarged/shrinked. — Self Similarity.

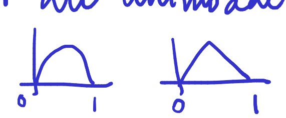
This is a property of fractal structures.

objects with fractional dimensions
(e.g. If a region of size l contains points proportional to l^α, where $\alpha \notin \mathbb{Z}$.)

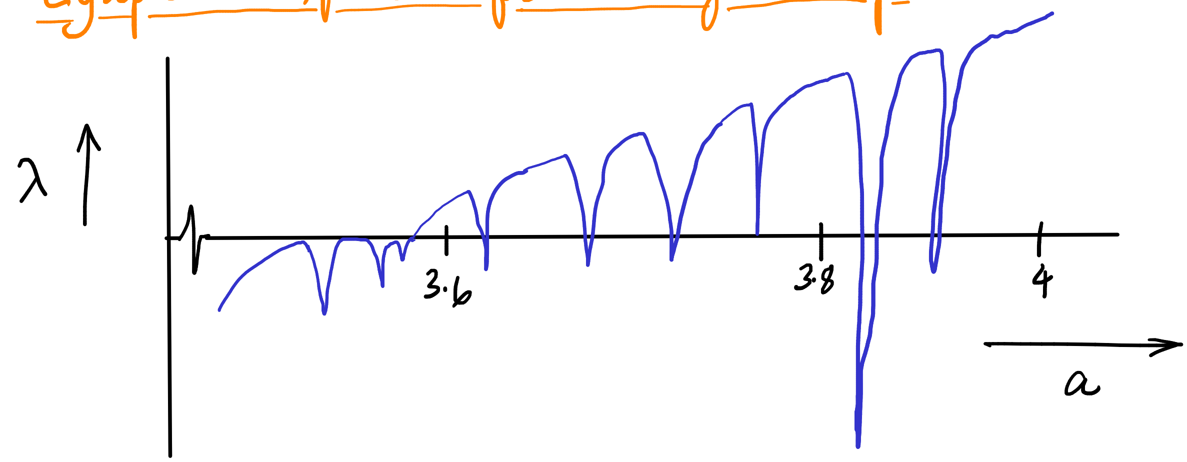
Figenbaum number, $\delta = \lim_{n \to \infty} \dfrac{a_n - a_{n-1}}{a_{n+1} - a_n} \simeq 4.669$

a_n : birth of 2^n cycle

This is universal for all unimodal distributions.



Lyapunov Exponent for the Logistic map:



Consider two initial conditions — one starting at x_0 and the other starting at $x_0+\delta_0$.

After n iterations, their values differ by δ_n.

$$|\delta_n| \simeq |\delta_0| e^{n\lambda}$$

$\lambda =$ Lyapunov Exponent

$\lambda > 0 \Rightarrow$ Chaos

$$\Rightarrow \lambda \simeq \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right|$$

$$= \frac{1}{n} \ln \left| \frac{f^n(x_0+\delta_0) - f^n(x_0)}{\delta_0} \right|$$ where $x_{n+1} = f(x_n)$

$$\simeq \frac{1}{n} \ln \left| (f^n)'(x_0) \right|$$

$f^2(x) = f(f(x))$,

$f^3(x) = f(f(f(x)))$, ...

$$(f^n)'(x_0) = \frac{d}{dx_0} f^n(x_0)$$

$$= \frac{d}{dx_0} f\left(f^{n-1}(x)\right) = \frac{d}{dx_0} f(x_{n-1}) = f'(x_{n-1}) \frac{dx_{n-1}}{dx_0}$$

$$= f'(x_{n-1}) \left(f^{n-1}\right)'(x_0) = \ldots$$

$$= f'(x_{n-1}) f'(x_{n-2}) \ldots f'(x_0)$$

$$\Rightarrow \quad \lambda \simeq \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| f'(x_i) \right|$$

$$\lambda \equiv \lim_{n\to\infty} \left(\frac{1}{n} \sum_{i=0}^{n-1} \ln \left| f'(x_i) \right| \right)$$

For the logistic map, $\quad f(x) = ax(1-x)$

$$\Rightarrow f'(x) = a - 2ax$$

$$\Rightarrow \lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| a - 2ax \right|$$