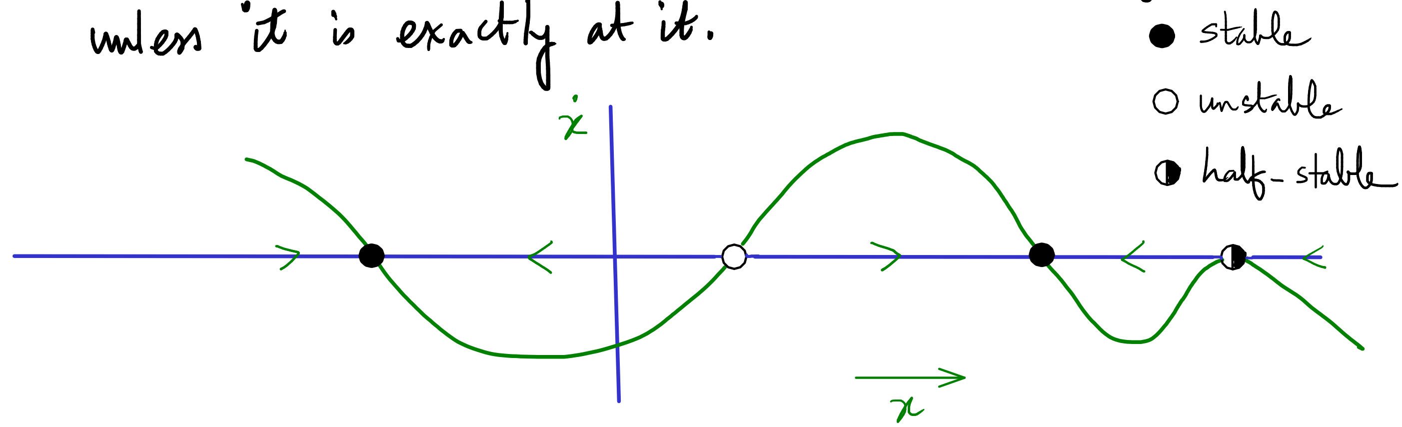
One dimensional flows: $\dot{x} = f(x)$

- If $f(x_0) = 0$, then x_0 is a fixed point
- The flow at x is towards the right if $f(x) > 0$ and is towards the left if $f(x) < 0$.
- A fixed point is stable if x approaches it when it is close to it and is unstable if x moves away from it unless it is exactly at it.



In-class assignments:

Find the fixed points for the following 1D flows and characterize their stability.

1. $\dot{x} = x^2 - 1$
2. Charging of RC circuit
3. $\dot{x} = e^x - \cos x$ (qualitative, using graph)
4. $\dot{x} = ax(b-x)$ $\quad a, b > 0$

Examples of two-dimensional fixed points:

① $\dot{x} = x(5-x-3y)$

$\dot{y} = y(3-x-y)$

Lotka-Volterra model
(Interaction of 2 species)

Fixed points:

1. $x=0,\ y=0$ $\quad (0,0)$
2. $x=0,\ 3-x-y=0$ $\quad (0,3)$
3. $y=0,\ 5-x-3y=0$ $\quad (5,0)$
4. $5-x-3y=0,\ 3-x-y=0$ $\quad (2,1)$

The stability of the fixed points can be understood using "Linear Stability Analysis" around each of the fixed points. Read about it from Strogatz's book.

② Pendulum: $\ddot{\theta} + \omega_0^2 \sin\theta = 0$

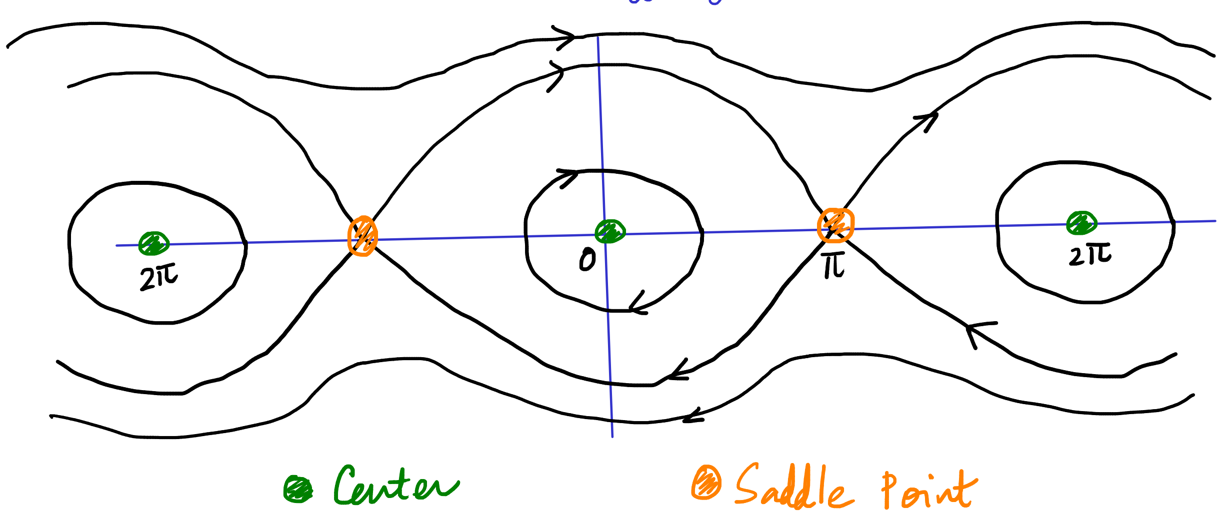
Equivalent system of first order ODEs:

$\dot{\theta} = \omega$

$\dot{\omega} = -\omega_0^2 \sin\theta$

Fixed points: $\theta = n\pi,\ n \in \mathbb{Z}$

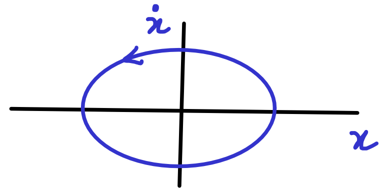
$\omega = 0$

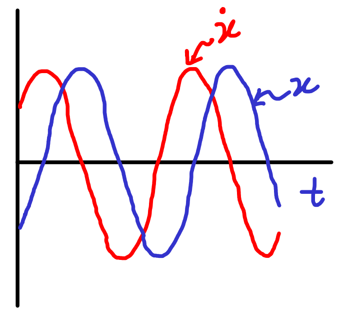


Example of a limit cycle

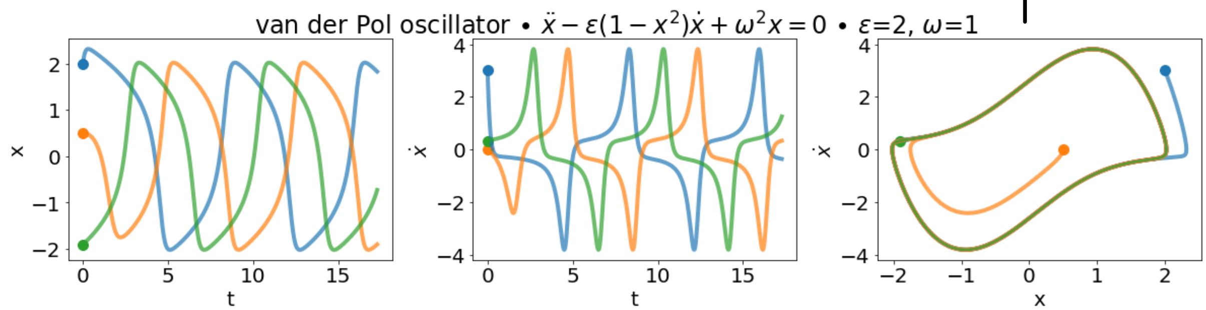
van der Pol oscillator $\quad m\ddot{x} - \epsilon(1-x^2)\dot{x} + m\omega^2 x = 0 \ , \ \epsilon > 0$

$\epsilon = 0 : \quad \ddot{x} + \omega^2 x = 0$





$\epsilon \neq 0$



Strange attractors and Chaotic trajectories

A strange attractor is an attractor that is an extensive irregularly shaped region in phase space. It has fractal geometry. Motion on a strange attractor appears random. Trajectories in the strange attractor explore regions of the attractor repeatedly, without exactly passing through the same point twice.

Chaotic trajectories show the following properties.

1. Mixing → Consider any two arbitrarily small regions in the attractor. A trajectory that passes through the first region must pass through the other region.
2. Quasiperiodicity → Trajectories pass through the attractor repeatedly and irregularly without ever closing on to itself, i.e., without any fixed time period.
3. Dense → Trajectories pass arbitrarily close to every point

⇒ Motion on a strange attractor exhibit ergodicity

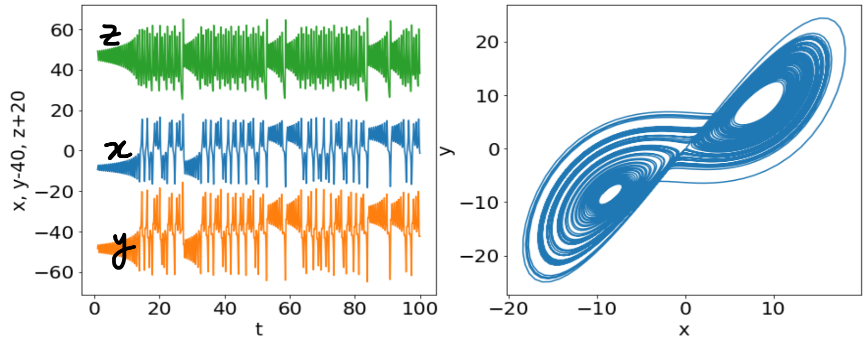
Strange attractor for the Lorenz system

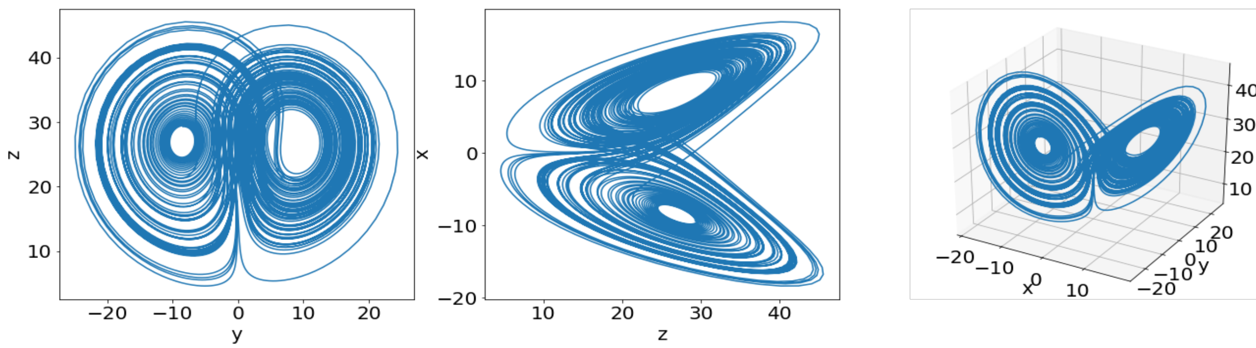
$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

$\sigma = 10$

$\rho = 28$

$\beta = 8/3$





Sensitivity to initial conditions

Small change in initial conditions $\Rightarrow$ large change in phase space coordinates after finite time.

E.g. an elliptic Kepler orbit may become parabolic.

The KAM theorem is valid for small perturbations. As the amount of perturbations increases, chaotic behavior may set in.

For a spaceship orbiting the earth, a small rocket boost may result in an orbit with a larger "radius", but a large boost may throw the spaceship off its orbit.

Fluids $\rightarrow$ As the fluid speed increases, streamline flow may become turbulent.

two nearby points get arbitrarily far from each other, on average, after finite time

two nearby points remain close, on average, after finite time

Lyapunov Exponent

[alternate spellings $L\begin{pmatrix} i \\ y \\ j \end{pmatrix}apuno\begin{pmatrix} v \\ ff \\ w \end{pmatrix}$]

Separation between two orbits after time, t, is,

$$S(t) \sim S_0 e^{\lambda t}, \quad \lambda = \text{Lyapunov exponent}$$

This is valid for $S(t) \ll$ Size of the attractor.

$\lambda > 0 \Rightarrow$ Chaos

$\lambda < 0 \Rightarrow$ Approach to a regular attractor.

Solar system $\Rightarrow$ For mercury, numerical estimates give an estimate, $\lambda \sim 3 \times 10^{-10}/\text{year}$ for the Lyapunov exponent.

$\Rightarrow$ Mercury might hit Venus or leave its orbit after about

$\tau \sim 3.5 \times 10^{9}$ years $= 3.5$ Gyr.

Motion of asteroids are also chaotic.
