CLASSICAL CHAOS

Syllabus:
1. Periodic motions and perturbations;
2. Attractors;
3. Chaotic trajectories and Liapunov exponents;
4. The logistic equation.

Textbook: Goldstein, Poole, Safko -- Classical Mechanics

Chaos – Sensitive dependence on initial conditions

Poincare – Kolmogorov – Lorenz – ...

"Can the flap of a butterfly's wings in Brazil set off a tornado in Texas?"

Related topics – Turbulence, Thermalization, ...

Integrable systems → If a system with n degrees of freedom has n independent conserved quantities, the system is integrable.

Non-integrable systems

→ Systems that can be broken up into an intgrable part and a non-integrable perturbation and can be treated using perturbation theory

→ Systems that are not of above class but have solutions that are well-behaved – small change in initial conditions ⇒ Small difference in trajectory at later times

→ Chaotic systems – sensitive dependence on initial conditions. Small difference in initial conditions ⇒ large difference after a finite time interval.

Chaotic $\neq$ Random / Stochastic

A chaotic system is deterministic but appears random because we cannot provide initial conditions with arbitrarily high precision. Any small error gets magnified enormously.

Periodic Motion

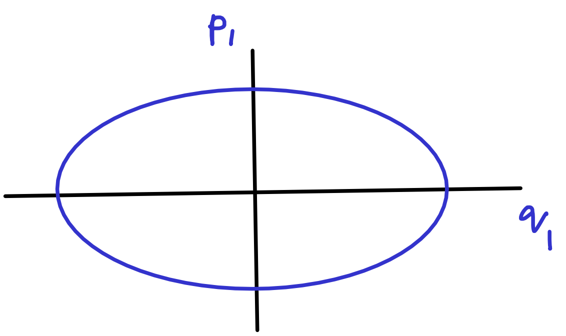
Examples – (1) Simple Harmonic Motion

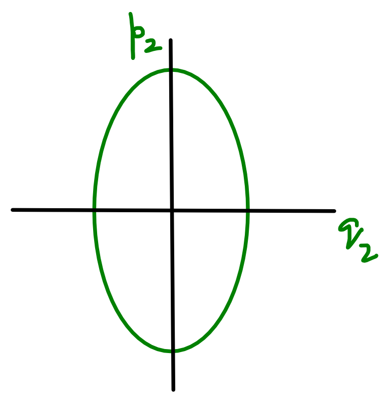
(2) Closed Kepler orbits

Dense periodic/quasiperiodic orbits

Consider uncoupled harmonic oscillators

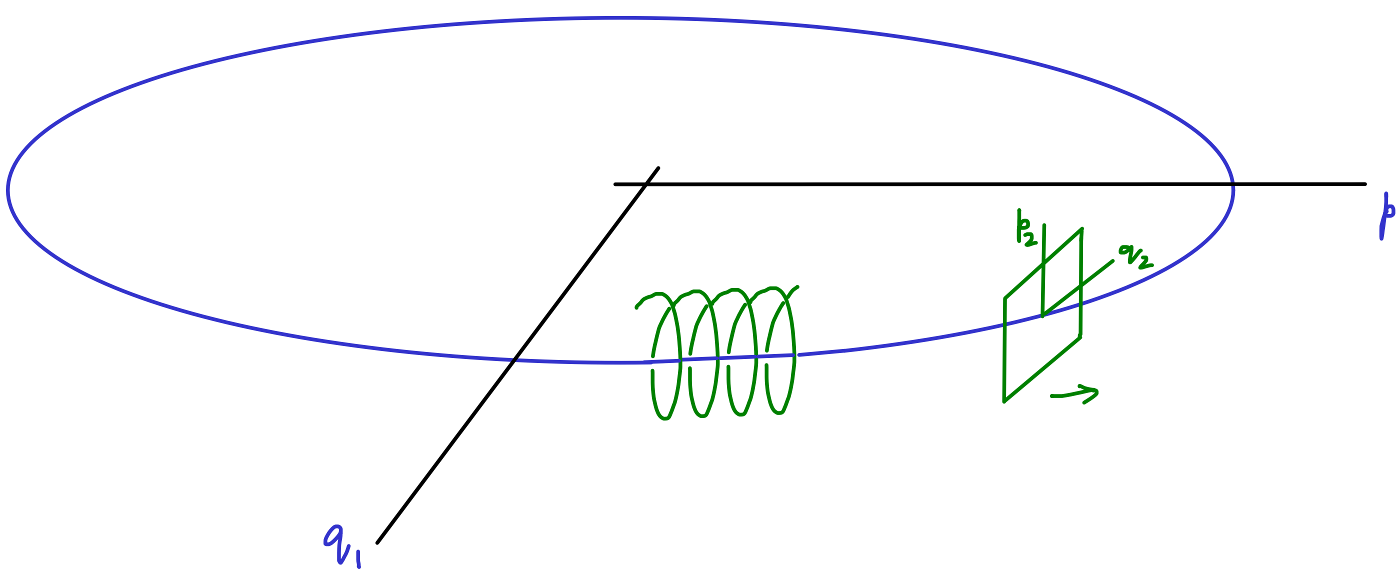
$$H = \frac{p_1^2}{2m_1} + \frac{1}{2} m_1 \omega_1^2 q_1^2 + \frac{p_2^2}{2m_2} + \frac{1}{2} m_2 \omega_2^2 q_2^2$$

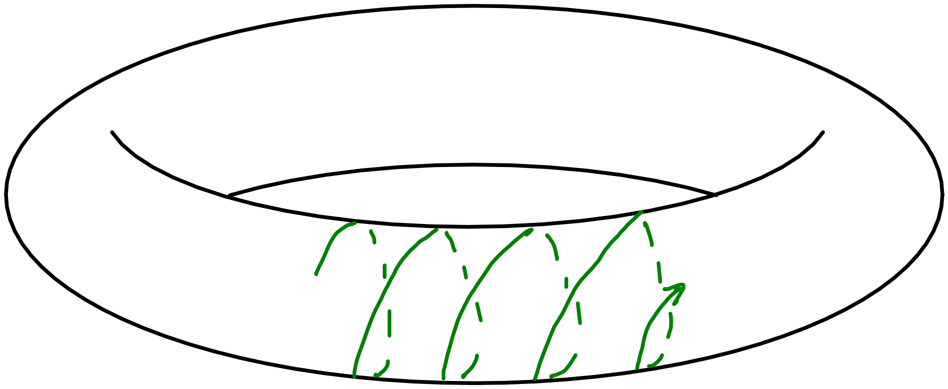




$$\frac{p_1^2}{2m_1} + \frac{1}{2} m_1 \omega_1^2 q_1^2 = E_1$$

$$\frac{p_2^2}{2m_1} + \frac{1}{2} m_2 \omega_2^2 q_2^2 = E_2$$





If $\frac{\omega_1}{\omega_2}$ is a rational number, then the system returns to its initial point in phase space after $LCM(2\pi/\omega_1, 2\pi/\omega_2)$ and repeats its motion.

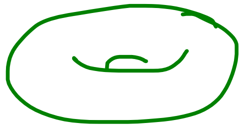
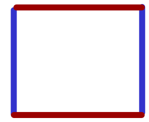
If $\frac{\omega_1}{\omega_2}$ is irrational, the trajectory does not close but comes arbitrarily close to its initial point.
↳ Dense periodic (quasiperiodic) orbit – bounded & confined to the surface of a torus.
Frequencies which are irrational multiples of each other said to be incommensurate frequencies.

Perturbations and the KAM theorem

Consider an integrable Hamiltonian $\mathcal{H}_0$, e.g., Earth + Sun system
Introduce an additional interaction that makes the system non-integrable, $\Delta\mathcal{H}$, e.g., Mars.

$\mathcal{H} = \mathcal{H}_0 + \Delta\mathcal{H}$

Kolmogorov-Arnold-Moser theorem:
If $\Delta\mathcal{H}$ is small and the frequencies describing motion under $\mathcal{H}_0$ are incommensurate, then, except for a negligible set of initial conditions, the motion under $\mathcal{H} = \mathcal{H}_0 + \Delta\mathcal{H}$ remains confined to an N-torus.

[1-torus : circle,
2-torus :  OR  (rectangle with periodic boundary conditions)]

The exceptions may be thought of as analogous to singular points in solutions of differential equations.

↳ When perturbations are small, most quasiperiodic orbits experience minimal change.

↳ In the presence of Mars, the Earth's orbit stays stable, and located near the same region, but its shape gets slightly modified.

Attractors

A Hamiltonian system when started with a certain initial condition, may evolve towards a fixed point in phase space or towards a stable orbit. These are examples of attractors.

An attractor is a set of points in phase space where a system reaches after transients die out.

A fixed point is a zero-dimensional attractor.

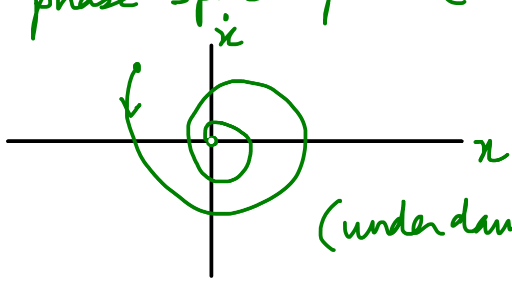
An orbit (also called limit cycle) is a one-dimensional attractor.

A strange attractor is a set of widely dispersed points in phase space and has fractal dimensions (non-integer dimensions).

Example of a fixed point attractor

Damped harmonic oscillator : $m\ddot{x} + b\dot{x} + m\omega^2 x = 0$

The phase space point $(x=0, \dot{x}=0)$ is a fixed point attractor.



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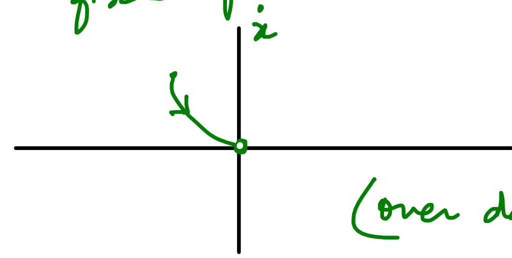

(over damped)