Special Integrals – Beta and Gamma Functions

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Syllabus (part of PHSDSC405T: Mathematical Methods II)

- Beta and Gamma Functions and Relation between them
- Expression of Integrals in terms of Gamma Functions
- Error Function (Probability Integral)

References
– J. Mathews and R. L. Walker, *Mathematical Methods of Physics*
– V. Balakrishnan, *Mathematical Physics*

I. GAMMA AND BETA FUNCTIONS

The *gamma function* is the solution to the *difference equation* (also known as an *iterative map*),

$$\Gamma(z) = (z-1)\Gamma(z-1), \tag{1}$$

with $\Gamma(1) = 1$.

[$\Gamma(z)$ is *analytic* in the entire complex plane, except for its simple poles at $\{0, -1, -2, -3, \dots\}$. $\dfrac{1}{\Gamma(z)}$ is an *entire function*. The meaning of these terms will become clear after your complex analysis course. You may ignore these for now.]

For Re $z > 0$, it can be expressed by the integral,

$$\boxed{\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx} \tag{2}$$

For this class, we will focus only on this formula. It is easy to check from Eq. 2 that $\Gamma(1) = 1$. Also using integration by parts, we see that Eq. 1 holds true. Thus, for whole numbers, $\Gamma(n+1) = n!$.

What is the value of $\Gamma\left(\frac{1}{2}\right)$?

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx \tag{3}$$

We substitute $x = \frac{u^2}{2} \Rightarrow dx = udu$.

$$\Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{2}\int_0^\infty e^{-u^2/2} du = \frac{\sqrt{2}}{2}\int_{-\infty}^\infty e^{-u^2/2} du = \frac{1}{\sqrt{2}}\sqrt{2\pi} = \sqrt{\pi} \tag{4}$$

The last step should be familiar from the probability class and comes from the normalization condition for a Gaussian distribution. This was derived by multiplying two copies of the integral $I = \int_{-\infty}^\infty e^{-u^2/2} du$ with the dummy integration variable as x and y. Thus,

$$I^2 = \int_{-\infty}^\infty e^{-x^2/2} dx \int_{-\infty}^\infty e^{-y^2/2} dy = \int_{-\infty}^\infty \int_{-\infty}^\infty dx\; dy\; e^{-(x^2+y^2)/2}$$

Going to polar coordinates, we get,

$$I^2 = \int_0^\infty \int_0^{2\pi} dr\; r d\theta\; e^{-r^2/2} = 2\pi.$$

Consider the product,

$$\Gamma(r)\Gamma(s) = \int_0^\infty \int_0^\infty dx\; dy\; x^{r-1} e^{-x} y^{s-1} e^{-y}.$$

Substituting, $x+y=u$, we get,

$$\Gamma(r)\Gamma(s) = \int_0^\infty du \int_0^u dx\ x^{r-1}(u-x)^{s-1}e^{-u}.$$

The upper limit of the x-integral can be understood in the following way. To cover the entire positive quadrant of the xy-plane, we may consider strips making 135° with the x-axis, i.e. parallel to lines having slope -1 ($x+y=$ constant). The locations of such strips are set by the values of the intercepts of these lines, denoted by u. For a given value of u, x can take values between $[0,u]$, since y must be non-negative.
Substituting, $x=ut$, we get,

$$\Gamma(r)\Gamma(s) = \int_0^\infty du\ x^{r+s-1}e^{-u} \int_0^1 dt\ t^{r-1}(1-t)^{s-1}$$

$$\Rightarrow \boxed{B(r,s) \equiv \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} = \int_0^1 dt\ t^{r-1}(1-t)^{s-1}.} \tag{5}$$

Here $B(r,s)$ denotes the *beta function*. It is easy to see that the beta function is symmetric in its arguments, i.e., $B(r,s)=B(s,r)$.
The integral form is valid only for Re $r,s>0$.
Reflection formula: It can be shown using complex contour integration that,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{6}$$

Substituting $t=\sin^2\theta$ in Eq. 5, we get, $B(r,s) = 2\int_0^{\pi/2} d\theta\ (\sin\theta)^{2r-1}(\cos\theta)^{2s-1}$.

II. ERROR FUNCTION

The *error function* is defined as,

$$\text{erf}(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt. \tag{7}$$

Check the following: $\text{erf}(\infty)=1$, $\text{erf}(-\infty)=-1$, $\text{erf}(0)=0$.
The error function is an odd function. It is a smooth step-like function which goes from -1 to +1 as we go from $-\infty$ to ∞ (qualitatively similar to the hyperbolic tangent function). The derivative of the error function is non-negative and approaches zero at $\pm\infty$ $\left(\text{erf}'(x) = \frac{2}{\sqrt{\pi}}e^{-x^2}\right)$.
Expanding the exponential, we get,

$$\begin{aligned}
\text{erf}(x) &= \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \\
&= \frac{2}{\sqrt{\pi}}\int_0^x \left(1 - t^2 - \frac{t^4}{2!} + \frac{t^6}{3!} - \cdots\right)dt \qquad (8)\\
&= \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} - \frac{x^5}{2!\times 5} + \frac{x^7}{3!\times 7} - \cdots\right). \qquad (9)
\end{aligned}$$

This series representation converges for all values of x.
The *complementary error function* is the function,

$$\text{erfc}(x) \equiv 1 - \text{erf}(x). \tag{10}$$

It is a monotonically decreasing step-like function which goes from 2 to 0 as its argument goes from $-\infty$ to ∞.
The cumulative standard normal (Gaussian) distribution is, given by,

$$P(Z<z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-u^2/2}du = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\int_0^z e^{-u^2/2}du = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_0^{z/\sqrt{2}} e^{-t^2}dt = \frac{1}{2}\left(1+\text{erf}\left(\frac{z}{\sqrt{2}}\right)\right) = \frac{1}{2}\text{erfc}\left(-\frac{z}{\sqrt{2}}\right)$$

Note:
(1) Related to the above discussion, if you are interested, you may read about Exponential Integrals, Sine Integrals, the Cosine Integrals, Fresnel Integrals (which will appear in your classes on diffraction) and Elliptic Integrals.
(2) You may consider reviewing these topics after your course on complex analysis.